Dcpo—Completion of posets

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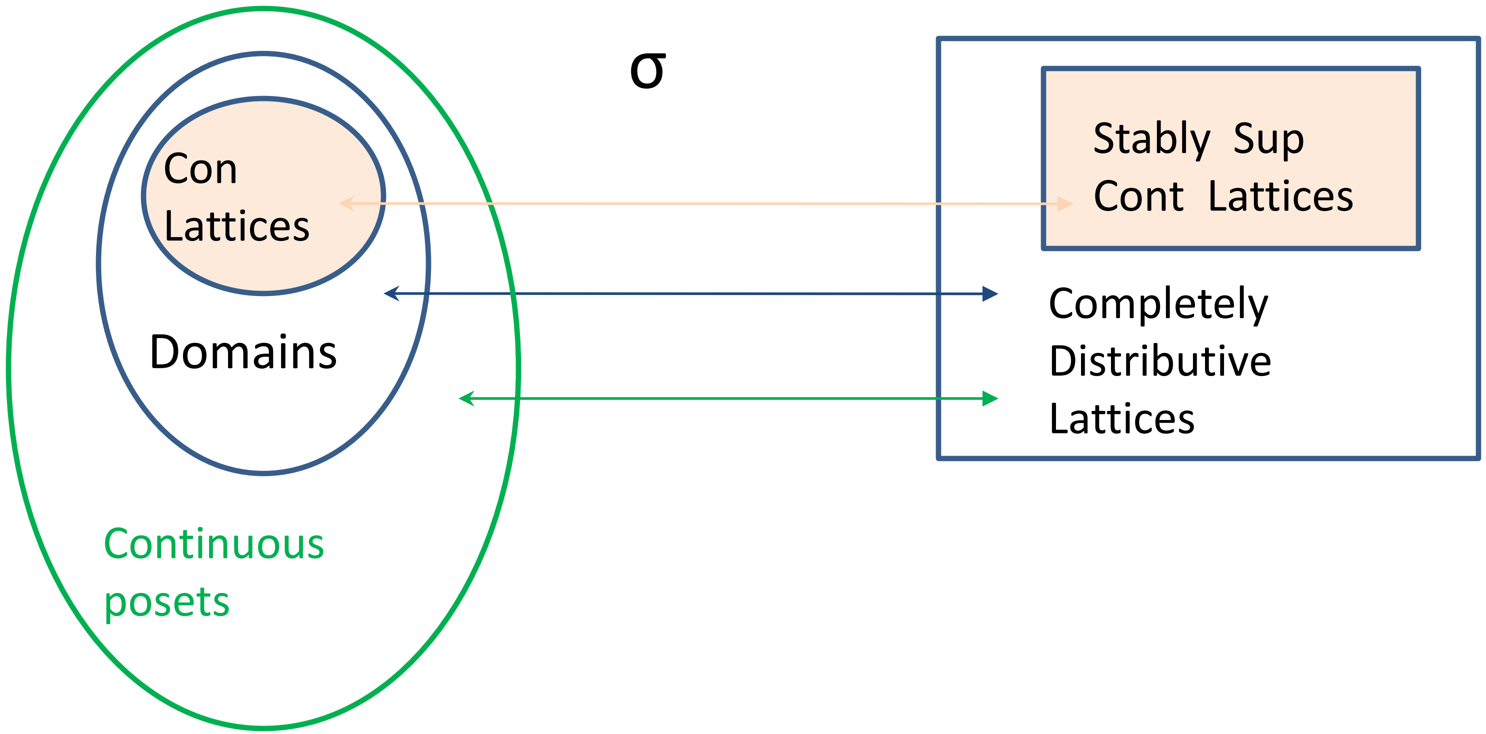
Outline

- Introduction
- D-completion of posets
- Properties
- Bounded complete dcpo completion
- Bounded sober spaces

1.Introduction

The Scott open (closed) set lattices:

For any poset P, let $\sigma(P)$ ($\sigma^{op}(P)$) be the complete lattice of all Scott open (closed) sets of P .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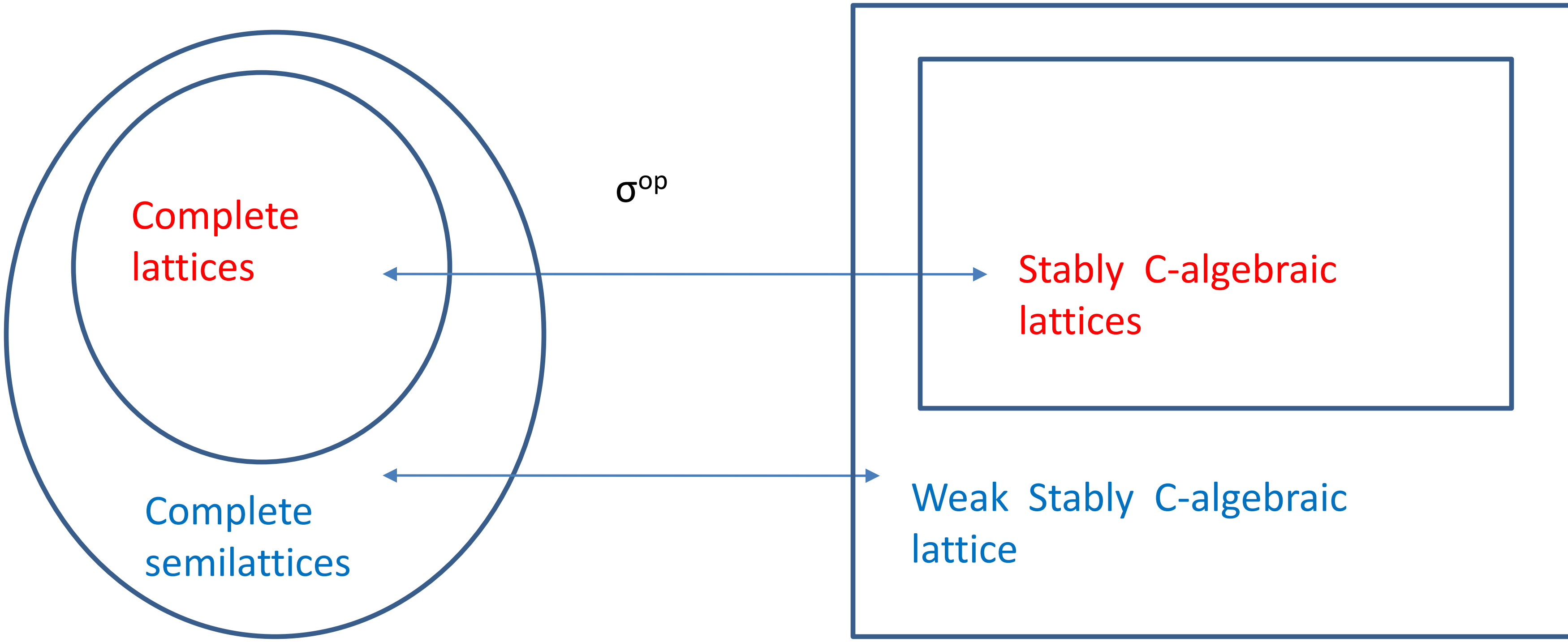
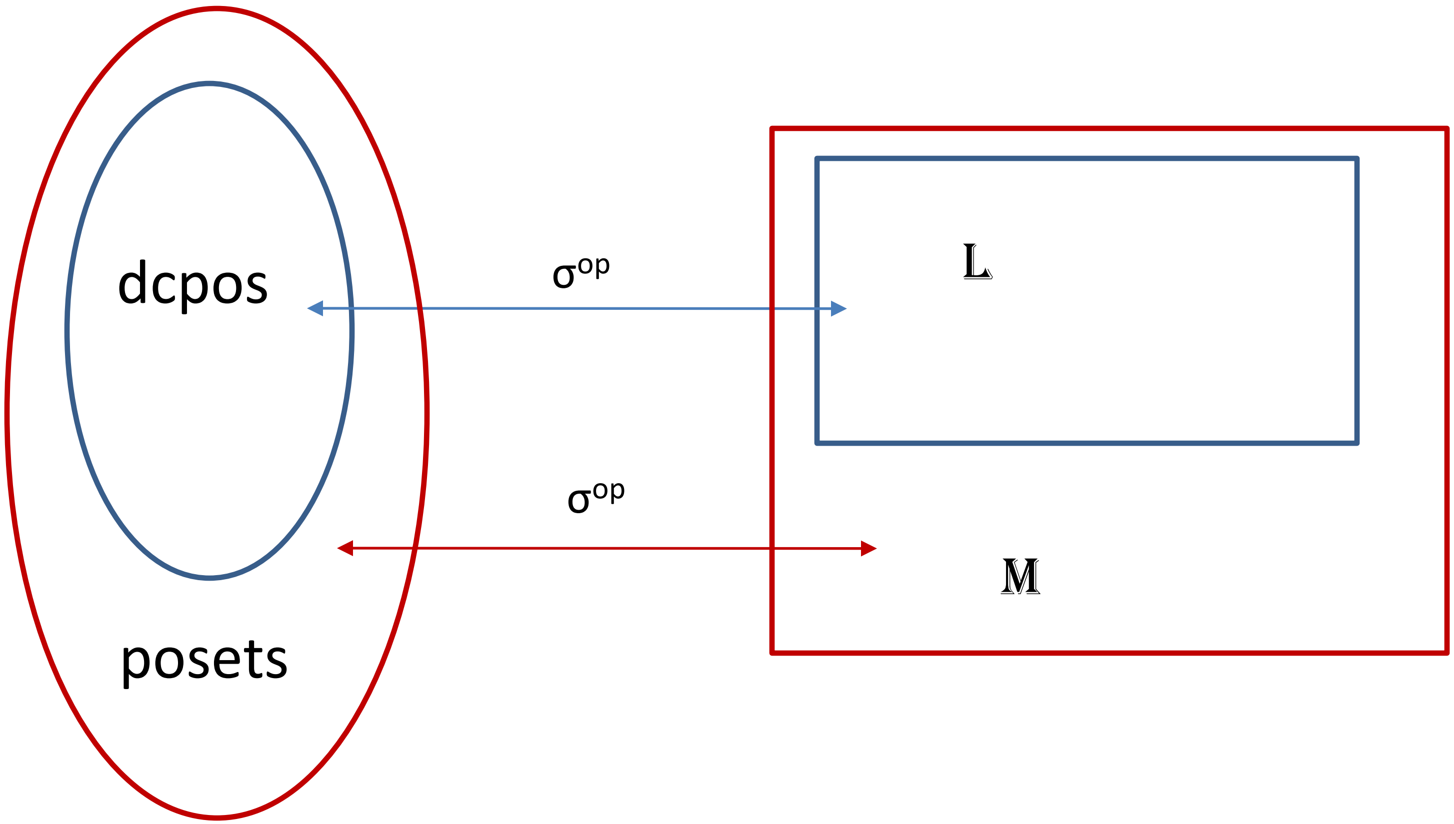
Lattices


σ^{op}
Complete lattices
Complete semilattices
Stably C-algebraic lattices
Weak Stably C-algebraic lattice



$\mathbf{M} = \mathbf{L}$?

2. Dcpo – completion of posets

Definition
Let P be a poset. The D – completion of P is a dcpo S together with a Scott continuous mapping $f: P \to S$ such that for any Scott continuous mapping $g: P \to Q$ from P into a dcpo Q, there is a unique Scott continuous mapping $g^*: S \to Q$ such that $g=g^*f$.

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, "g"'] & S \arrow[d, "g^*"] \\ & Q \end{tikzcd}$$

Every two D-completions of a poset are isomorphic !

Theorem 1.
For each poset P, the smallest subdcpo E(P) of σ^{op} (P) containing $\{ \downarrow x\!: x \in P \}$ is a D- completion of P. The universal Scott continuous mapping $\eta : P \to E(P)$ sends x in P to $\downarrow x$.

Subdcpo--- a subset of a dcpo that is closed under taking supremum of directed sets.

Let POS_d be the category of posets and the Scott continuous mappings.

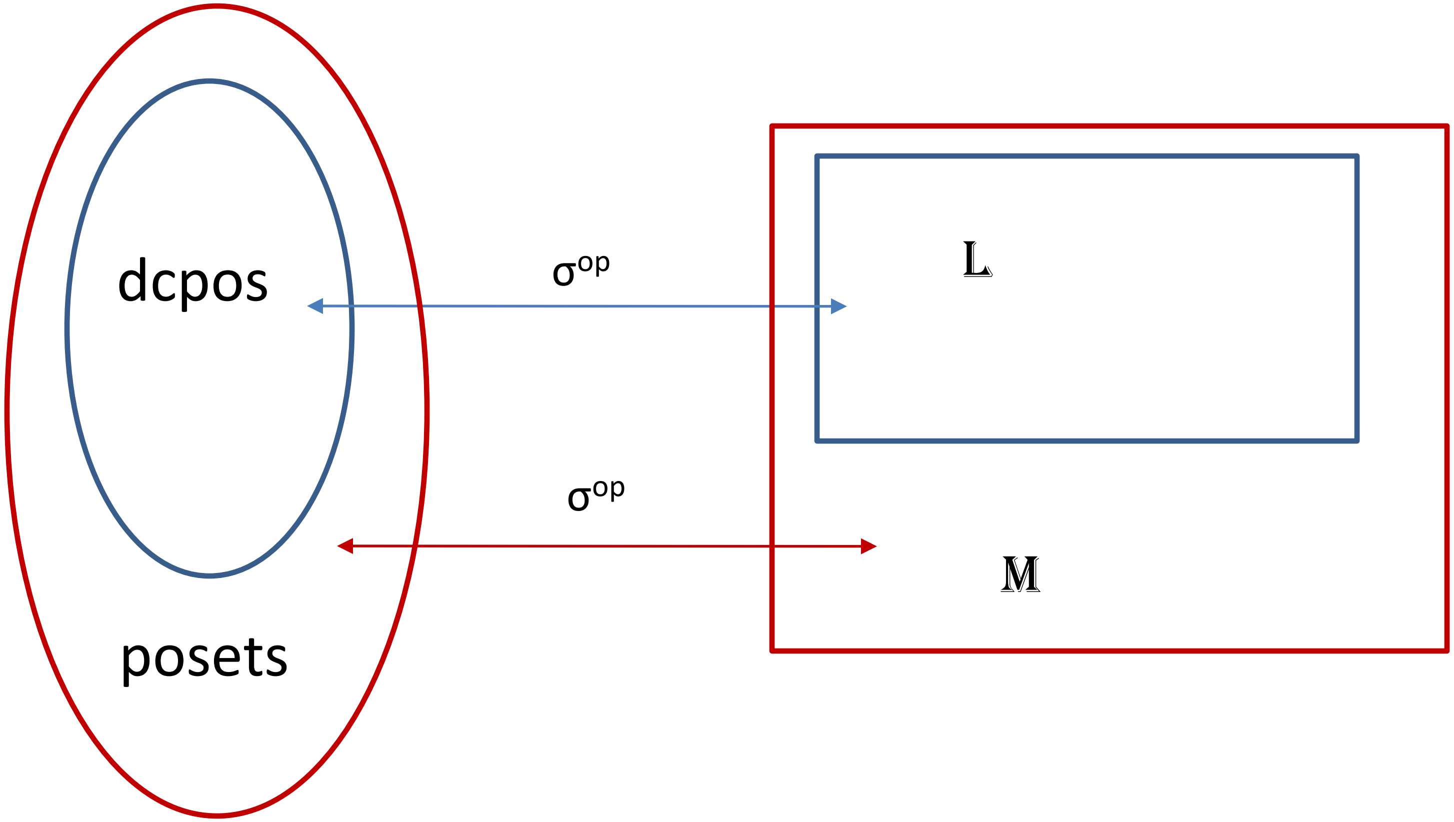
Corollary

The subcategory DCPO of POS_d consisting of dcpos is fully reflexive in POS_d .

2. Properties of D(P)

Theorem 2.
For any poset P,

$$\sigma^{op}(P) \cong \sigma^{op}(E(P)).$$



$\mathbf{M} = \mathbf{L}$

Theorem 3. A poset P is continuous if and only if E(P) is continuous.

Corollary If P is a continuous poset, then

$$E(P)=Spec(\sigma^{op}(P)).$$

Spec(L): The set of co-prime elements of L

Theorem 4 . A poset P is algebraic if and only if E(P) is algebraic.

3. Local dcpo-completion

Definition A poset P is called a local dcpo, if every upper bounded directed subset has a supremum in P.

Definition

Let P be a poset. A local dcpo completion of P is a local dcpo S together with a Scott continuous mapping f: $P \to S$ such that for any Scott continuous mapping g: $P \to Q$ from P into a local dcpo Q, there is a unique Scott continuous mapping g^*: $S \to Q$ such that g=g*f.

Theorem 5

For any poset P, the set BE(P) of all the members F of E(P) which has an upper bound in P and the mapping f: $P \longrightarrow$ BE(P), that sends x to $\downarrow x$, is a local dcpo completion of P.

Corollary

The subcategory LD of $\mathrm{POS_d}$ consisting of local dcpos is reflexive in $\mathrm{POS_d}$.

4. Bounded sober spaces

Definition A T_0 space X is called bounded sober if every non empty co-prime closed subset of X that is upper bounded in the specialization order * is the closure of a point .

* If it is contained in some cl{x}.

Example

If P is a continuous local dcpo, then ΣP is bounded sober.

For a T_0 space X, let

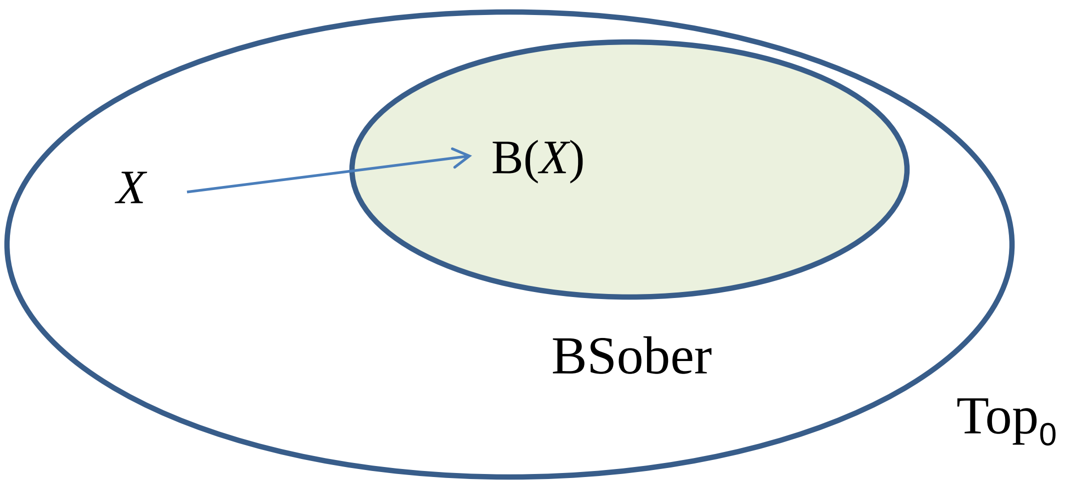
$B(X)=\{$ F: F is a co-prime closed set and is upper bounded in the specialization order $\}$.

For each open set U of X, let
$H_U = \{ F \varepsilon B(X):\ F \cap U \neq \emptyset \}$.
Then $\{H_U : U \varepsilon O(X)\}$ is topology on B(X).

Let B(X) denote this topological space.

Theorem 6

B(X) is the reflection of X in the subcategory BSober of bounded sobers spaces.



One question:

If P and Q are dcpos such that $\sigma(P)$ is isomorphic to $\sigma(Q)$, must P and Q be isomorphic?

THANK YOU!