Towards Sequential Synthetic Domain Theory

B. Reus (Univ. Sussex) and Thomas Streicher (TU Darmstadt)

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Aim of the talk

Synthetic Domain Theory (SDT) provides a logic for speaking about domains as particular sets.

Typical models are realizability toposes $\mathbf{RT}(\mathcal{A})$ ($\mathcal{A}$ a pca) together with a **dominance** true $\in \Sigma$.

SDT applies to **various** classes of domains. The general theory is well developed and also the particular case of *Domain Theory à la Scott*. But there are different models for it e.g. $\mathbf{RT}(K_2)$ or $\mathbf{RT}(\mathcal{P}\omega)$.

M. Hyland has suggested to consider/axiomatize domain theories of **different flavours**, e.g. stable, strongly stable . . . or **sequential**.

Previous Work

Longley and van Oosten have identified a pca $\mathcal{B} = [\mathbb{N} \rightharpoonup \mathbb{N}]$, a kind of "partial function realizability".

There is an obvious dominance Σ in $\mathbf{RT}(\mathcal{B})$. The ensuing category of domains contains strongly stable domains as a full subcategory.

The pca $\mathcal{B}$ is a universal object of the category $\mathcal{SA}$ of (countably based) concrete data structures and **sequential algorithms**.
One may consider $\mathcal{SA}$ also as a **typed pca** and one can show that $\mathbf{RT}(\mathcal{B}) \simeq \mathbf{RT}(\mathcal{SA})$.

Problem
What are good axioms for Σ in $\mathbf{RT}(\mathcal{SA})$?

Basic Ideas of Our Approach

Replace $\mathcal{SA}$ by $\mathcal{OSA}$, the wellpointed category of **observably sequential algorithms** (cells maybe filled with an error element $\top$).

$\mathcal{OSA}$ is equivalent to $\mathcal{LBD}$, Laird's category of **locally boolean domains**.

Since $\mathcal{SA}$ is a **lluf** subcat of $\mathcal{OSA}$ there is a universal object in $\mathcal{OSA}$, namely $\mathsf{U} = [\mathsf{N} \to \mathsf{N}]$ where $\mathsf{N} = \mathbb{N}_{\bot}^{\top}$.

Thus, by a theorem of Lietz and S.

$$\mathbf{RT}(\mathcal{OSA}) \simeq \mathbf{RT}(\mathcal{U}) \qquad \text{topos}$$

with $\mathcal{U}$ the (untyped) pca of global elements of U.

Refinement of the Dominance (1)

J. Laird's **key observation** is that in $(\mathcal{O})\mathcal{SA}$ the dominance Σ (one cell and one value) can be **decomposed** as

$$\Sigma \cong [0 \to 0]$$

where 0 has **one cell** and **no value**.

NB
In $\mathcal{OSA}$ type 0 has global elements $\bot$ and $\top$.
In $\mathcal{OSA}$ the type $\Sigma = [0 \to 0]$ has globale elements $\bot$, id and $\top$.
$\top$ **absent** in $\mathcal{SA}$.

Refinement of the Dominance (2)

$\mathbf{Asm}(\mathcal{OSA})$ and $\mathbf{Asm}(\mathcal{U})$ are the respective full subcats of $\mathbf{RT}(\mathcal{OSA})$ and $\mathbf{RT}(\mathcal{U})$ on $\neg\neg$-separated object.
For $A \in \mathcal{OSA}$ let $\Delta(A)$ — or simply A — be the projective modest set whose underlying set are the global elements of A, whose type of realizers is $A \in \mathcal{OSA}$ and $||a||_{\Delta(A)} = \{a\}$.

Although id $\in \Delta(\Sigma)$ is a dominance
$\top \in 0$ is **not** a dominance
since 0 has **no meet** operation $\wedge : 0 \times 0 \to 0$.
Still $\top \in 0$ classifies a pullback stable class of monos which, however, are not closed under composition.

Refinement of the Dominance (3)

We can recover Σ and Σ_c as subobjects of $[0 \to 0]$ as follows

$$\Sigma = \{f \in [0 \to 0] \mid f(\bot) = \bot\} \qquad \Sigma_c = \{f \in [0 \to 0] \mid f(\top) = \top\}$$

NB
(1) Inclusion into $[0 \to 0]$ are not split!
(2) Both Σ and Σ_c are closed under composition to be understood as $\wedge$ and $\vee$ respectively.

Reconstructing the Various Orderings

For $X \in \mathbf{RT}(\mathcal{OSA})$ we may define the orderings

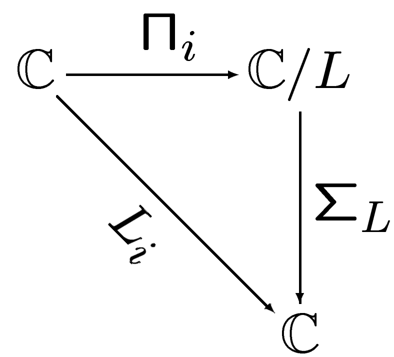
$$\begin{array}{lll} x \sqsubseteq y & \text{iff} & \forall p \in [X \to \mathsf{0}]\Big(p(x) = \top \Rightarrow p(y) = \top\Big) \\ x \leq_s y & \text{iff} & \exists f \in [\Sigma \to X]\Big(f(\bot) = x \wedge f(\mathsf{id}) = y\Big) \\ x \leq_c y & \text{iff} & \exists f \in [\Sigma_c \to X]\Big(f(\mathsf{id}) = x \wedge f(\top) = y\Big) \\ x \leq_b y & \text{iff} & \exists f \in [\mathsf{0} \to X]\Big(f(\bot) = x \wedge f(\top) = y\Big) \end{array}$$

Theorem
Recalling that $\mathcal{LBD} \simeq \mathcal{OSA}$ for $X \in \mathcal{LBD}$ the above orders on $\Delta(X)$ coincide with the *extensional*, *stable*, *costable* and *bistable* order on X, respectively.

J. Laird pointed out a mistake in a previous wrong variant.

Various (Bi)Liftings

In a lccc $\mathbb{C}$ every $i : 1 \to L$ induces a **lifting** operation



Instantiating i by the inclusion of id into $[0 \to 0]$, Σ and Σ_c, respectively, we get endofunctors

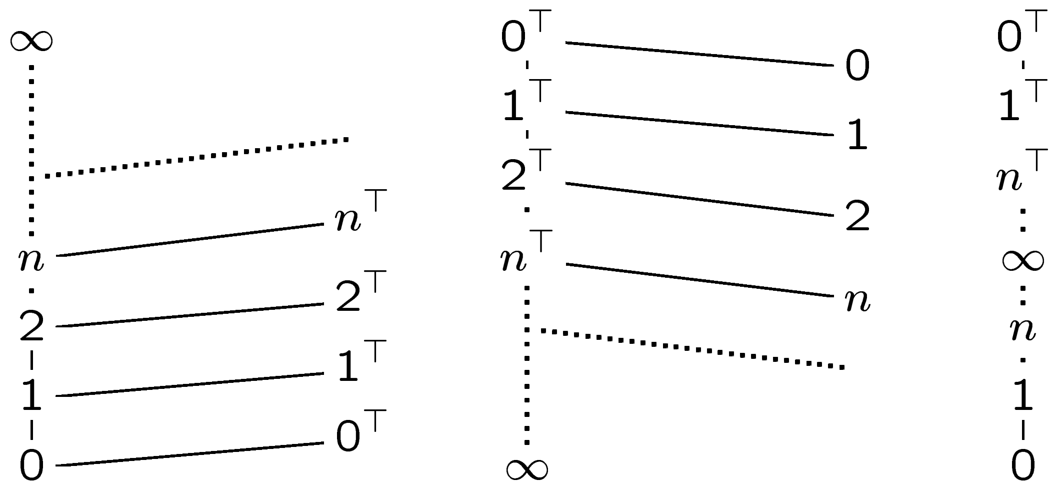
$$(-)_\perp^\top \qquad (-)_\perp \qquad (-)^\top$$

on $\mathrm{RT}(\mathcal{LBD})$ called **bilifting**, **lifting** and **colifting**, respectively.

Fixpoints of (Bi)Liftings

Theorem
All those functors admit final (and thus also initial) fixpoints inherited from the **bifree** fixpoint $\bar{\omega}$ of $(-)^{\top}_{\bot}$ in $\mathcal{LBD}$.

The stable, costable and extensional order of $\bar{\omega} \in \mathcal{LBD}$ look as follows



final fixpoint of $(-)_{\bot}$ obtained by removing the $n^{\top}$'s
final fixpoint of $(-)^{\top}$ obtained by removing the n's

Synthetic Construction of Fixpoints

$\bar{\omega}$ as the set of all $f : \mathbb{N} \to \mathsf{O}^{\mathsf{O}}$ such that

$$\forall j < i \, \forall u \in \mathsf{O} \left(f(i)(u) = u \Rightarrow f(j)(u) = u \right)$$

$\bar{\omega}_s = \{ f \in \bar{\omega} \mid \forall n \in \mathbb{N}. \, f(n) \in \Sigma \}$

$\bar{\omega}_c = \{ f \in \bar{\omega} \mid \forall n \in \mathbb{N}. \, f(n) \in \Sigma_c \}$

intitial algebras can be carved out as **least subalgebras**

Tentative Axioms (1)

We observe a few facts about or model and state them as axioms. Whether they are enough is necessarily an **empiric/pragmatic** issue since axiomatisations can hardly be categorical (pun intended!).

(Axiom 0) $\forall u, v \in \mathsf{O} \left(\neg\neg u = v \Rightarrow u = v\right)$

says that equality on O is $\neg\neg$-closed

(Axiom 1) $\forall u, v \in \mathsf{O}.\, ((\top = u) \Leftrightarrow (\top = v)) \Rightarrow u = v$

says that $\lambda u \in O.\,(\top = u) : \mathsf{O} \to \Omega$ is monic.

(Axiom 2) $\forall F \in \mathsf{O}^{\mathsf{O}^{\mathbb{N}}}.\, (F \neq \lambda f.\bot \wedge F \neq \lambda f.\top) \Rightarrow \exists n \in \mathbb{N}.\ F = \pi_n$

all non-constant $f : \mathsf{O}^{\mathbb{N}} \to \mathsf{O}$ are projections
(realized by the ω-ary catch of $\mathcal{LBD}$).

Tentative Axioms (2)

One can show that Σ is a subobject of Ω where $u \in \Sigma$ is identified with $(u(\top) = \top)$.

(Axiom 3) $\quad \forall p \in \Sigma . \forall q \in \Omega . \, (p \Rightarrow (q \in \Sigma)) \Rightarrow (p \wedge q) \in \Sigma$

says that Σ is a dominance.

(Axiom4) $\quad \forall F : \Sigma^{\mathbb{N}} \to \Sigma . \, (F(\lambda n . \top) = \top) \Rightarrow$
$$\neg\neg\exists K \in \mathcal{P}_f(\mathbb{N}) . \forall f \in \Sigma^{\mathbb{N}} . \, F(f) = \top \Leftrightarrow \forall k \in K . f(k) = \top$$

expresses continuity (*c.f.* Scott's axiom).

Tentative Axioms (3)

A variant of (Axiom 4) with $[0 \to 0]$ instead of Σ is

(Axiom5) $$\forall F : [0 \to 0]^{\mathbb{N}} \to 0.\ \neg\neg\exists\langle n_1, \ldots . n_k\rangle \in \mathbb{N}^*.\ \neg\neg\exists u \in 0.$$
$$\forall f \in [0 \to 0]^{\mathbb{N}}.\ F(f) = (f(n_1) \circ \cdots \circ f(n_k))(u)$$

Open Questions

- How are the notions of completeness, induced by the various inclusions of initial into final algebras, related?

- Are our axioms strong enough for showing that retracts of $\mathsf{U} = [\mathbb{N}_\perp^\top \to \mathbb{N}_\perp^\top]$ are closed under the various type formers?