On Variants of Modified Bar Recursion

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(Joint work with Martín Escardó)

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Outline

1. Background
2. Three Realizability Bar Recursions
   - BBC bar recursion
   - Berger's bar recursion
   - Escardo's bar recursion

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Interpretions of Arithmetic and Analysis

$$
\begin{tikzcd}
\mathsf{PA}^\omega \arrow[r, "N\text{-}trans"] & \mathsf{HA}^\omega \arrow[r, bend left, "Dialec."] \arrow[r, bend right, "A\text{-}trans{+}mr"'] & \mathbf{T}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \mathbf{T} + \mathsf{SBR} \\
\mathsf{PA}^\omega + \mathsf{AC}_{0,\rho} \arrow[r, "N\text{-}trans"] & \mathsf{HA}^\omega + \mathsf{AC}^N_{0,\rho} \arrow[ur, "Dialec."] \arrow[dr, "A\text{-}trans{+}mr"'] & \\
 & & \mathbf{T} + \mathsf{MBR}
\end{tikzcd}
$$

Primitive recursion vs Bar recursion

$$\mathsf{R}(n) \stackrel{\mathcal{T}}{=} \begin{cases} G & \text{if } n = 0 \\ H_n(\mathsf{R}(n-1)) & \text{otherwise} \end{cases}$$

$$\mathsf{SBR}(s^{\rho^*}) \stackrel{\mathcal{T}}{=} \begin{cases} G_s & \text{if } Y(\hat{s}) < |s| \\ H_s(\lambda x^\rho.\mathsf{SBR}(s * x)) & \text{otherwise.} \end{cases}$$

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The Challenge

$$\forall n \exists x A(n,x) \rightarrow \exists f \forall n A(n, fn)$$

The Challenge

$$\forall n \exists x A(n,x) \to \exists f \forall n A(n, fn)$$

$$\forall n \neg\neg\exists x A^N(n,x) \to \neg\neg\exists f \forall n A^N(n, fn)$$

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$$\forall n \neg\neg\exists x A^N(n,x) \to \neg\neg\exists f \forall n A^N(n, fn)$$

$$\forall n((\exists x A^B(n,x) \to B) \to B) \wedge (\exists f \forall n A^B(n, fn) \to B) \to B$$

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Given realizers for

$$\forall n ((\exists x A^B(n,x) \to B) \to B)$$

$$\exists f \forall n A^B(n, fn) \to B$$

produce realizer for B.

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Given realizers for

$$\forall n((A(n) \to B) \to A(n))$$

$$\forall n A(n) \to B$$

produce realizer for B.

The Challenge

Given

$$\begin{aligned} H_n &: (A(n) \to B) \to A(n) \\ Y &: \forall n A(n) \to B \end{aligned}$$

Produce a realiser for B (or $\forall n A(n)$).

The Challenge

Given

$$\begin{array}{lcl} H_n & : & (A(n) \to B) \to A(n) \\ Y & : & \forall n A(n) \to B \end{array}$$

Produce a realiser for B (or $\forall n A(n)$).

Sketch of solution: (assume $s^{(\mathbb{N}\times\rho)^*} \; : \; \forall n \in s \, A(n)$)

$$\Psi(s)(n) \stackrel{\rho}{=} \begin{cases} s(n) & \text{if } n \in s \\ H_n(\lambda x^\rho . Y(\Psi(s * \langle n, x \rangle))) & \text{otherwise.} \end{cases}$$

Berardi, Bezem, Coquand (BBC) functional

$$\Psi(s) = s \,@\, \lambda n.H_n(\lambda x.Y(\Psi(s * \langle n, x\rangle)))$$

- Efficient
- **Not** easy to prove total
- **Not** easy to prove it is a realiser

Berger's observation

Enough: Given

$$\begin{aligned} H_n &: (A(k) \to B) \to A(n) \\ Y &: \forall n A(n) \to B \end{aligned}$$

Produce a realiser for $\forall n A(n)$.

Berger's observation

Enough: Given

$$\begin{array}{lll} H_n & : & (A(k) \to B) \to A(n) \\ Y & : & \forall n A(n) \to B \end{array}$$

Produce a realiser for $\forall n A(n)$.

Sketch of solution: (assume $s^{\rho^*} \; : \; \forall n < |s| \, A(n)$)

$$\Psi(s)(n) \stackrel{\rho}{=} \begin{cases} s_n & \text{if } n < |s| \\ H_n(\lambda x^\rho . Y(\Psi(s * x))) & \text{otherwise.} \end{cases}$$

Berger's (MBR) functional

$$\Psi(s) = s \,@\, \lambda n.H_n(\lambda x.Y(\Psi(s * \langle |s|, x\rangle)))$$

- **Not** very efficient
- Easy to prove total
  (*by bar induction*)
- Easy to prove it is a realiser
  (*by bar induction*)

Question

Can we solve the original general problem

efficiently with an **easy proof** of correctness?

Escardo's trick

Given

$$\begin{aligned} H_n &: (A(n) \to B) \to A(n) \\ Y &: \forall n A(n) \to B \end{aligned}$$

Produce a realiser for $\forall n A(n)$.

Sketch of solution: (assume $s^{\rho^*} : \forall n < |s| \, A(n)$)

$$\Psi(s)(n) \stackrel{\rho}{=} \begin{cases} s_n & \text{if } n < |s| \\ H_n(\lambda x^\rho . Y(\Psi(s * \langle n, x \rangle))) & \text{otherwise.} \end{cases}$$

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Produce a realiser for $\forall n A(n)$.

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$$\Psi(s)(n) \stackrel{\rho}{=} \begin{cases} s_n & \text{if } n < |s| \\ H_n(\lambda x^\rho . Y(\Psi(s * \ldots * \langle n, x \rangle))) & \text{otherwise.} \end{cases}$$

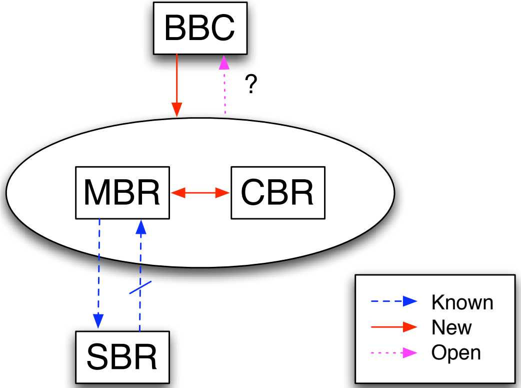
where $\ldots \equiv \Psi(s)[|s|, n-1]$.

Escardo's (CBR) bar recursion

$$\Psi(s) = s\,@\,\lambda n.H_n(\lambda x.Y(\Psi(\overline{\Psi(s)}(n) * \langle n, x\rangle)))$$

- Efficient
- Easy to prove total
  (*by course-of-value bar induction*)
- Easy to prove it is a realiser
  (*by course-of-value bar induction*)

Main results



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  Berger and Oliva, MSCS, 16:163–183, 2006

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