Completeness for Moss's Coalgebraic Logic (Boolean version)

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coalgebras

coalgebra $X \to TX$ we have for every T a notion of T-bisimilarity

T : $\mathsf{Set} \to \mathsf{Set}$ *weak-pullback preserving* functor

Paradimatic example: $T = \mathcal{P}$ (powerset-functor: coalgebras are Kripke frames)

Other examples: Labelling of states and transitions (input and output), deterministic automata, probabilistic transition systems, stochastic transtition systems, arbitrary combinations of these: infinitely many examples

Non-example: neighbourhood frames ($TX = 2^{2^X}$) are coalgebras for a non weak-pullback-preserving functor

weak pullback preserving functors ...

... lift from Set to Rel (sets with relations as arrows),
ie, for each $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ we have $\bar{T} : \mathsf{Rel} \longrightarrow \mathsf{Rel}$.

A relation is a span, to which we can apply T

$$
\begin{tikzcd}
& R \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& TR \arrow[dl] \arrow[dr] & \\
TX & & TY
\end{tikzcd}
\qquad
\begin{tikzcd}
& TR \arrow[d] \arrow[ddl] \arrow[ddr] & \\
& \bar{T}R \arrow[dl] \arrow[dr] & \\
TX & & TY
\end{tikzcd}
$$

$\bar{T}$ is a functor iff T preserves weak pullbacks

Moss's coalgebraic logic

Given T

The language $\mathcal{L}$ is closed under Boolean operations and

if $\alpha \in T_\omega \mathcal{L}$ then $\nabla \alpha \in \mathcal{L}$

T_ω is the finitary version of T, technically, $T_\omega X = \bigcup\{TY \mid Y \subseteq_\omega X\}$.

Example: $\mathcal{P}_\omega X$ is the set of all *finite* subsets of X

$$x \Vdash \nabla \alpha \;\Leftrightarrow\; (\xi(x), \alpha) \in \bar{T}(\Vdash)$$

Example ($T = \mathcal{P}$): Moss's logic is equi-expressive with the basic modal logic:

$$x \Vdash \nabla \phi \;\Leftrightarrow\; x \Vdash \Box \bigvee \phi \wedge \{\bigwedge \Diamond a \mid a \in \phi\}$$

Thm(Moss): $\mathcal{L}$ is invariant under bisimulation. The original version with infinitary conjunctions (no other Booleans needed) characterises bisimilarity (Hennessy-Milner property).

algebraic reformulation of the semantics

$$\begin{tikzcd} & \in \arrow[dl] \arrow[dr] & \\ X & & \mathcal{P}X \end{tikzcd} \quad \text{lifted} \quad \begin{tikzcd} & \bar{T}(\in) = \bar{\in} \arrow[dl] \arrow[dr] & \\ TX & & T\mathcal{P}X \end{tikzcd} \quad \text{is} \quad \mathcal{P}TX \xleftarrow{\rho_X} T\mathcal{P}X$$

Define $L = FT_\omega U$ (where $U : \mathsf{BA} \to \mathsf{Set}$ and F its left-adjoint).

$\rho : TPX \to PTX$ induces a BA-morphism $LPX \to PTX$.

The semantics of $\mathcal{L}$ wrt $\xi : X \to TX$ is given by the 'complex algebra' of X:

$$\begin{tikzcd} LPX \arrow[r] & PTX \arrow[r, "P\xi = \xi^{-1}"] & PX \\ L\mathcal{L} \arrow[u, "L[\![-]\!]"] \arrow[rr] & & \mathcal{L} \arrow[u, "[\![-]\!]"'] \end{tikzcd}$$

[ρ is the semantics of ∇]

$$x \Vdash \nabla\alpha \Leftrightarrow \xi(x) \in \rho_X(\alpha)$$

Notation: $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$, $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$, $\mathbb{P} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{BA}$

examples

$$\begin{tikzcd} & \in \arrow[dl] \arrow[dr] & \\ X & & \mathcal{P}X \end{tikzcd} \quad \text{lifted} \quad \begin{tikzcd} & \bar{T}(\in) = \bar{\in} \arrow[dl] \arrow[dr] & \\ TX & & T\mathcal{P}X \end{tikzcd} \quad \text{is} \quad \mathcal{P}TX \xleftarrow{\rho_X} T\mathcal{P}X$$

Examples for $\alpha \bar{\in} \Phi$ or $\alpha \in \rho_X(\Phi)$:

$T = \mathcal{P}$: $\forall x \in \alpha . \exists \phi \in \Phi . x \in \phi$ and vice versa

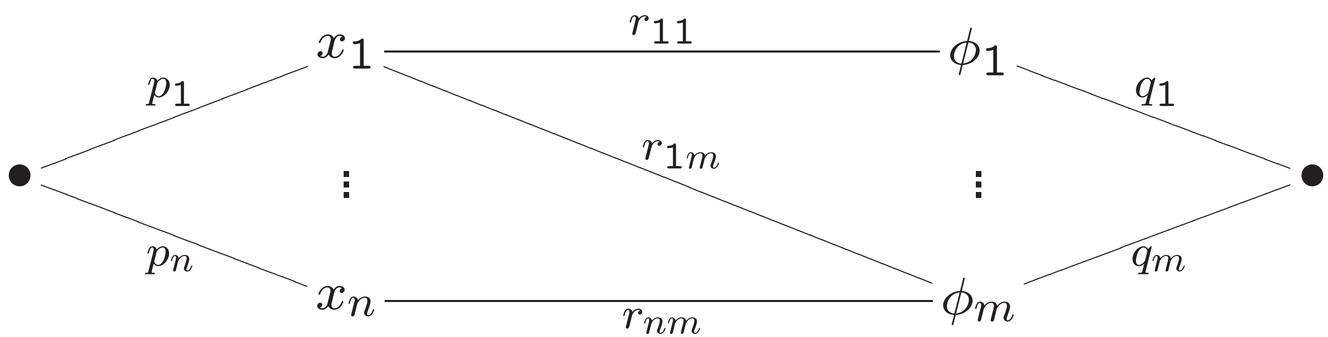
$$\begin{tikzcd} x_1 \arrow[rr, dash] \arrow[ddrr, dash] & & \phi_1 \\ \vdots & & \vdots \\ x_n \arrow[rr, dash] & & \phi_m \end{tikzcd}$$

examples

$$\begin{tikzcd}[column sep=small] & \in \arrow[dl] \arrow[dr] & \\ X & & \mathcal{P}X \end{tikzcd}$$ lifted $$\begin{tikzcd}[column sep=small] & \bar{T}(\in) = \bar{\in} \arrow[dl] \arrow[dr] & \\ TX & & T\mathcal{P}X \end{tikzcd}$$ is $\mathcal{P}TX \xleftarrow{\rho_X} T\mathcal{P}X$

Examples for $\alpha \bar{\in} \Phi$ or $\alpha \in \rho_X(\Phi)$:

$$TX = \{d : X \to [0,1] \mid d(x) = 0 \text{ almost everywhere } \}$$



the proof system (T restricts to finite sets)

Notation:

$\mathcal{L}$	$a, b, c, \ldots$	$T_\omega \mathcal{L}$	$\alpha, \beta, \gamma \ldots$
$\mathcal{P}_\omega \mathcal{L}$	$\phi, \psi, \ldots$	$T_\omega \mathcal{P}_\omega \mathcal{L}$	$\Phi, \Psi, \ldots$
$\mathcal{P}_\omega T_\omega \mathcal{L}$	$A, B, C \ldots$		

If T preserves finite sets (maps finite sets to finite sets):

(∇1)	From $\alpha \overline{\preceq} \beta$ infer $\vdash \nabla\alpha \preceq \nabla\beta$
(∇2)	$\bigwedge\{\nabla\alpha \mid \alpha \in A\} \preceq \bigvee\{\nabla(T\bigwedge)(\Phi) \mid \Phi \in SRD(A)\}$
(∇3)	$\nabla(T\bigvee)(\Phi) \preceq \bigvee\{\nabla\alpha \mid \alpha \overline{\in} \Phi\}$

$$(\nabla 2) \quad \bigwedge\{\nabla\alpha \mid \alpha \in A\} = \bigvee\{\nabla(T\bigwedge)(\Phi) \mid \Phi \in SRD(A)\}$$

Remark: This axiom is important: it allows do eliminate conjunctions (and the essence of the completeness proof will be to show that every $\mathcal{L}$-formula is interderivable with a conjunction free normal form). This has repercussions, eg, in the modal μ-calculus where alternating automata are equivalent to non-deterministic automata.

$$\rho_X : TPX \to PTX$$

Example: $A = \{\alpha, \beta\} \in PTX$, $T = \mathcal{P}$, $\alpha = \{a_1, a_2\}, \beta = \{b_1, b_2\}$

What can we say about $\nabla\alpha \wedge \nabla\beta$?

$\Phi \in SRD(A)$ iff $\Phi \in TPX$ such that $\rho(\Phi) \supseteq A$

the proof system (general case: infinitary rules)

For an arbitrary (weak pullback preserving) functor $T : \mathsf{Set} \to \mathsf{Set}$

$$(\nabla 1) \quad \frac{\{b_1 \preceq b_2 \mid (b_1, b_2) \in Z\}}{\nabla\alpha \preceq \nabla\beta} \, (\alpha, \beta) \in \overline{Z}$$

$$(\nabla 2) \quad \frac{\{\nabla(T \bigwedge)(\Phi) \preceq a \mid \Phi \in SRD(A)\}}{\bigwedge\{\nabla\alpha \mid \alpha \in A\} \preceq a}$$

$$(\nabla 3) \quad \frac{\{\nabla\alpha \preceq a \mid \alpha \overline{\in} \Phi\}}{\nabla(T \bigvee)(\Phi) \preceq a}$$

reminder: completeness of the basic modal logic **K**

(in the style of Domain Theory in Logical Form)

$$\begin{tikzcd} \mathsf{BA} \arrow[r, bend right, "S"'] & \mathsf{Set} \arrow[l, bend right, "\mathbb{P}"'] \end{tikzcd}$$

Define $\mathbb{K}$: BA $\to$ BA as follows:

$\mathbb{K}(\mathbb{A})$ is generated by $\Box a$, $a \in \mathbb{A}$, modulo $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box\top = \top$.

Note: Every 'variable' a, b is under the scope of exactly one modality

Thm: $\mathbb{K}\mathbb{P}X \to \mathbb{P}\mathcal{P}X$, $\Box a \mapsto \{b \subseteq a\}$, is an isomorphism for finite sets X.

Cor: a) One-step completeness: $\mathbb{K}\mathbb{P}X \to \mathbb{P}\mathcal{P}X$ is injective for all X.
b) Completeness of **K**.

remark on ‘one-step completeness’

(Now writing T for $\mathcal{P}$)

Show

$\mathbb{KP}X \to \mathbb{P}TX$ injective (completeness via normal form)

or

$TX \to S\mathbb{KP}X$ surjective (completeness via building a satisfying model)

$\mathbb{M}$ and the one-step proof system

What is the analog of $\mathbb{K}$ in our case?

Define $\mathbb{M}$: BA $\longrightarrow$ BA

$\mathbb{M}\mathbb{A}$ is given by

generators: $\nabla\alpha$, $\alpha \in T_\omega U\mathbb{A}$

modulo: $(\nabla 1)$-$(\nabla 3)$

In the paper we make precise what we mean by 'modulo' here: we call it the *one-step proof system*

final coalgebra sequence and initial algebra sequence

$$1 \longleftarrow \mathcal{P}1 \longleftarrow \cdots \longleftarrow \mathcal{P}^n 1 \longleftarrow \cdots \longleftarrow \mathcal{P}^\omega 1 \qquad \text{canonical model}$$

$$\mathbb{2} \longrightarrow \mathbb{K}\mathbb{2} \longrightarrow \cdots \longrightarrow \mathbb{K}^n \mathbb{2} \longrightarrow \cdots \longrightarrow \mathbb{K}^\omega \mathbb{2} \qquad \text{Lindenbaum algebra}$$

[two references: Abramsky’89, Ghilardi’95]

final coalgebra sequence and initial algebra sequence

$$1 \longleftarrow \mathcal{P}1 \longleftarrow \cdots \longleftarrow \mathcal{P}^n 1 \longleftarrow \cdots \longleftarrow \mathcal{P}^\omega 1 \qquad \text{canonical model}$$

$$\mathbb{2} \longrightarrow \mathbb{K}\mathbb{2} \longrightarrow \cdots \longrightarrow \mathbb{K}^n \mathbb{2} \longrightarrow \cdots \longrightarrow \mathbb{K}^\omega \mathbb{2} \qquad \text{Lindenbaum algebra}$$

$$\mathbb{P}1 \longrightarrow \mathbb{P}\mathcal{P}1 \longrightarrow \cdots \longrightarrow \mathbb{P}\mathcal{P}^n 1 \longrightarrow \cdots \longrightarrow \mathbb{P}\mathcal{P}^\omega 1 \qquad \text{canonical extension}$$

final coalgebra sequence and initial algebra sequence

$$
\begin{tikzcd}
1 & \mathcal{P}1 \arrow[l] & \cdots \arrow[l] & \mathcal{P}^n 1 \arrow[l] & \cdots \arrow[l] & \mathcal{P}^\omega 1 \arrow[l] & \text{canonical model}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{2} \arrow[r] \arrow[d, "\cong" description] & \mathbb{K}\mathbb{2} \arrow[r] \arrow[d, "\cong" description] & \cdots \arrow[r] & \mathbb{K}^n \mathbb{2} \arrow[r] \arrow[d, "\cong" description] & \cdots \arrow[r] & \mathbb{K}^\omega \mathbb{2} \arrow[d, tail] & \text{Lindenbaum algebra} \\
\mathbb{P}1 \arrow[r] & \mathbb{P}\mathcal{P}1 \arrow[r] & \cdots \arrow[r] & \mathbb{P}\mathcal{P}^n 1 \arrow[r] & \cdots \arrow[r] & \mathbb{P}\mathcal{P}^\omega 1 & \text{canonical extension}
\end{tikzcd}
$$

from one-step completeness to completeness

$\mathcal{L}$: Moss's language, $\mathcal{L}_i$: formulas of depth i

$$\begin{tikzcd}
\mathcal{L}_0 \arrow[r] \arrow[d] & \mathcal{L}_1 \arrow[r] \arrow[d] & \mathcal{L}_2 \arrow[d] & \cdots & \mathcal{L} \arrow[d] \\
\mathcal{L}_0/\equiv \arrow[r] \arrow[d] & \mathcal{L}_1/\equiv \arrow[r] \arrow[d] & \mathcal{L}_2/\equiv \arrow[d] & \cdots & \mathcal{L}/\equiv \arrow[d] \\
\mathbb{2} \arrow[r] \arrow[d] & \mathbb{M}\mathbb{2} \arrow[r] \arrow[d] & \mathbb{M}^2\mathbb{2} \arrow[d] & \cdots & \mathcal{M} \arrow[d] \\
\mathbb{P}1 \arrow[r] & \mathbb{P}T1 \arrow[r] & \mathbb{P}T^2 1 & \cdots & \mathbb{P}T^\omega 1
\end{tikzcd}$$

from one-step completeness to completeness needs ...

$\mathcal{L}_n/\equiv \longrightarrow \mathbb{M}^n 2$ is iso [Derivations of $\vdash a \equiv b$ of terms a, b of depth n can be performed without using terms of depth $> n$. Follows from the fact that the logic is described by a one-step proof system.]

$\mathbb{M}$ is a functor [Given a BA-morphism $f : \mathbb{A} \to \mathbb{B}$, a derivation of $a \equiv a'$ in the one-step proof system over $\mathbb{A}$ can be mapped to a derivation of $f(a) \equiv f(a')$ in the one-step proof system over $\mathbb{B}$.]

$\mathbb{M}$ is finitary and preserves embeddings [Given an injective BA-morphism $f : \mathbb{A} \to \mathbb{B}$, a derivation of $f(a) \equiv f(a')$ in the one-step proof system over $\mathbb{B}$ can be mapped to a derivation of $a \equiv a'$ in the one-step proof system over $\mathbb{A}$ (proof uses that for a finite BA $\mathbb{A}$ an embedding $\mathbb{A} \to \mathbb{B}$ has a half-inverse (which follows eg from the fact that complete Boolean algebras are injective))]

one-step completeness

Show $\delta : \mathbb{MP}X \to \mathbb{P}TX$ is injective. Idea: Find a half-inverse.

How can we go from $\mathbb{P}TX$ to $\mathbb{MP}X$? Recall: $\mathbb{MP}X$ is genereated by elements in TPX (and $\mathbb{P}TX$ is generated by elements in TX).

So we need $TX \to TPX$, which is provided by applying T to $\{\} : X \to PX$.

So let $\mathcal{G} = \{\nabla(T\{\}(\alpha) \mid \alpha \in T_\omega X\}$. Note that $\delta(\nabla(T\{\}(\alpha)) = \{\alpha\}$.

We have to show $\forall a \in \mathbb{MP}X.\ a = \bigvee\{\nabla\beta \in \mathcal{G} \mid \nabla\beta \leq a\}$.

Case 1: $a = \nabla\beta$, $\beta \in T_\omega PX$. Uses $(\nabla 3)$: $\nabla(T\bigvee)(\Phi) \preceq \bigvee\{\nabla\alpha \mid \alpha \overline{\in} \Phi\}$

Case 2: $a = \neg\nabla\beta$. Uses $(\nabla 4)$: From $\vdash \top \preceq \bigvee \phi$ infer $\vdash \top \preceq \bigvee\{\nabla\alpha \mid \alpha \in T\phi\}$

Case 3: $a = \bigwedge \beta_i$. Uses $(\nabla 2)$: $\bigwedge\{\nabla\alpha \mid \alpha \in A\} \preceq \bigvee\{\nabla(T\bigwedge)(\Phi) \mid \Phi \in SRD(A)\}$

conclusion

Given a category $\mathcal{X}$ and a functor $T : \mathcal{X} \to \mathcal{X}$, what can we say about logics for T-coalgebras? [in the talk: $\mathcal{X} = \mathsf{Set}$.]

What is the propositional 'base logic'? Choose a category $\mathcal{A}$ of algebras with appropriate $P : \mathcal{X} \to \mathcal{A}$ [in the talk: $\mathcal{A} = \mathsf{BA}$, P powerset.]

Extend the base logic by modal operators and axioms: choose a functor $L : \mathsf{BA} \to \mathsf{BA}$ and semantics $\delta : LP \to PT$ [δ inuduces map $\mathsf{Coalg}(T) \to \mathsf{Alg}(L)$.]

One of the strength of this approach is that it is parametric in the base categories. For example, we want to look at (future work):

$\mathcal{A}$ could be distributive lattices, complete atomic Boolean algebras

$\mathcal{X}$ could be posets