The Exotic, Arcane, and Plain In the Formal Ball Domain

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The Formal Ball Domain

For a metric space (X, d), we define the formal ball domain $\mathbf{B}^+X$ as the set $X \times \mathbb{R}^+$ ordered by

$$(x, r) \sqsubseteq (y, s) \text{ if } d(x, y) \leq r - s.$$

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The approximation relation $\ll$ in $\mathbf{B}^+X$ is given by

$$(x, r) \ll (y, s) \text{ iff } d(x, y) < r - s,$$

from which it readily follows that $\mathbf{B}^+X$ is a continuous poset or predomain.

Normed Spaces

In a normed vector space E, the correspondence

$$(x, r) \leftrightarrow \overline{B}_r(x) := \{y | d(x, y) \leq r\}$$

defines an order-isomorphism between the formal ball domain $\mathbf{B}^+E$ and the set of closed balls of radius ≥ 0 ordered by reverse inclusion. This holds in particular for euclidean space $\mathbb{R}^n$.

Domain Environments

The embedding $x \mapsto (x, 0)$ is a topological embedding onto the maximal points of the formal ball domain endowed with the relative Scott topology. Such representations of topological spaces are frequently called domain environments or domain representations or computational models for the space.

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The formal ball domain as a computational model was introduced by K. Weihrauch and U. Schreiber (1981) and much more systematically investigated by A. Edalat and R. Heckmann (1998).

Topology vs. Order

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Sets of the form $D \times Q$, where D is dense in X and Q is dense in $\mathbb{R}^+$, form approximating bases (in the sense of domain theory) for $\mathbf{B}^+X$. Hence X is separable metric iff $\mathbf{B}^+X$ is countably based.

Completeness

The formal ball domain $\mathbf{B}^+X$ is directed complete (and hence a domain=continuous dcpo) iff (X, d) is a complete metric space.

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In the formal ball domain we see the first strong interplay between domain theoretic and topological notions of completeness, a theme of continuing research interest for domain representations of more general spaces than metric.

Topological and Domain Constructions

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Theorem. *For X complete metric, the Plotkin power domain of the formal ball domain B^+X is a domain environment for the space of compact subsets with the Vietoris (or Hausdorff metric) topology.*

Theorem.*For X a separable complete metric space, the probabilistic power domain of B^+X is a domain environment for the space of probability measures on X (with the usual weak topology).*

The Extended Formal Ball Domain

By introducing formal balls of negative radius we obtain the extended formal ball domain $\mathbf{B}X = X \times \mathbb{R}$, with order and approximating relation given by the same formulas as those for $\mathbf{B}^{+}X$.

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One of the beauties of the extended formal ball domain is the existence of the order-reversing involutions such as

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Each slice $X \times \{r\}$ is a copy of X (in the relative Scott topology), and (X, d) is complete iff $\mathbf{B}X$ is conditionally directed complete.

Normed Spaces Again

For a normed vector space E with closed unit ball B,

$$C := \{(rx, -r) \in E \oplus \mathbb{R} : x \in B,\ 0 \leq r\}$$

is a closed proper cone in the product topological vector space with conal base $B \times \{-1\}$, a copy of B. (For B the unit ball in $\mathbb{R}^2$, we obtain the usual circular cone in $\mathbb{R}^3$ opening around the negative z-axis.)

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Identifying $E \oplus \mathbb{R}$ with $E \times \mathbb{R}$, the extended formal ball domain, it turns out that the formal ball domain order agrees with the conal order, i.e., $(x, r) \sqsubseteq (y, s) \Leftrightarrow (y, s) - (x, r) = (y - x, s - r) \in C$.

D-completions

Recall that a monotone convergence space is a T_0-space in which a directed set in the order of specialization converges to its supremum. For a T_0-space X, the D-completion is the reflection X^d of X into category of monotone convergence spaces (O. Wyler (1981), Y. Ershov (1997), K. Keimel & J. L. (2007)). For predomains equipped with the Scott topology it agrees with the rounded ideal completion equipped with the Scott topology.

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If a metric space X is embedded in its completion $\tilde{X}$, then the D-completion of $X \times (r, \infty) \subseteq \tilde{X} \times \mathbb{R} = \mathbf{B}X$ is $\tilde{X} \times [r, \infty)$. In particular, the D-completion of $\mathbf{B}^+X$ is $\mathbf{B}^+\tilde{X}$.

The conditional D-completion of $X \times \mathbb{Q}$ is $\tilde{X} \times \mathbb{R}$, which is also the dual conditional D-completion.

FS-Domains

Recall that a continuous dcpo is called an FS-domain if the identity map is a directed sup of finitely separated continuous self-maps. The FS-domains form a maximal cartesian closed category of domains.

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Theorem. (J.L., 2007) *Let X be a complete metric space for which each closed ball is totally bounded. Then $\mathrm{B}^+X_\perp$, the domain of formal balls with a bottom element adjoined, is an FS-domain.*

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Theorem. (J.L., 2007) *Let X be a complete metric space for which each closed ball is totally bounded. Then $\mathrm{B}^+X_\perp$, the domain of formal balls with a bottom element adjoined, is an FS-domain.*

This result holds in particular for the domain of closed balls for $\mathbb{R}^n$. It is an open question whether these domains are retracts of bifinite domains.

The Product Topology of $\mathbf{B}X$

For a metric space X, there are close, but delicate, connections between the product topology on $X \times \mathbb{R} = \mathbf{B}X$ and standard domain topologies.

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Theorem. *An upper set in $\mathrm{B}X$ is Scott open iff it is open in the product topology. Such sets and their order duals form a subbasis for the product topology, and hence the biScott topology and the product topology agree.*

The Hyperbolic Topology

The hyperbolic topology of a metric space (X, d) is the topology with subbasic open sets of the form

$$\{z : d(z, x) - d(z, y) < t\} \text{ for } x, y \in X,\ t \in \mathbb{R}.$$

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Theorem. (Y. Hattori and H. Tsuiki (2007)) *The hyperbolic topology on X agrees with the metric topology iff the product topology of $\mathbf{B}X = X \times \mathbb{R}$ agrees with the Lawson topology.*

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Example. The hyperbolic and norm topologies disagree in ℓ_1 and agree in ℓ_p for $1 < p < \infty$.

References

A. Edalat and R. Heckmann, A computational model for metric spaces, *Theoret. Comput. Sci.* 193 (1998), 53–73.

Yu. L. Ershov, On d-spaces, *T.C.S.*, 224 (1999), 59–72.

K. Keimel and J. Lawson, D-completions and the d-topology, *T.C.S.* (to appear).

J. Lawson, Metric spaces and FS-domains, *T.C.S.* 405 (2008), 73–74.

H. Tsuiki and Y. Hattori, Lawson topology of the space of formal balls and the hyperbolic topology of a metric space, *T.C.S.* (to appear).

K. Weihrauch and U. Schreiber, Embedding metric spaces into cpo's, *T.C.S.* 16 (1981), 5–24.

O. Wyler, Dedekind complete posets and Scott topologies, *Springer Lecture Notes on Mathematics* 871 (1981), 384–389.