A Continuous Dcpo Representation of Regular Formal Topologies

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Formal topology includes a constructive theory of domains, in the sense that the formal space $Pt(\mathcal{X})$ of a Scott (or unary) formal topology $\mathcal{X}$ is a Scott domain when ordered by inclusion, and every Scott domain arises in this way (G. Sambin, 1987).
Background/Motivation

- Formal topology includes a constructive theory of domains, in the sense that the formal space $Pt(\mathcal{X})$ of a Scott (or unary) formal topology $\mathcal{X}$ is a Scott domain when ordered by inclusion, and every Scott domain arises in this way (G. Sambin, 1987).

- In fact, the formal space $Pt(\mathcal{X})$ of $\mathcal{X}$ can be embedded in a Scott domain $D = (Pt(\mathcal{X}_S), \subseteq)$, where $\mathcal{X}_S$ is the Scott compactification of $\mathcal{X}$ (so $\mathcal{X}_S$ is Scott) (G. Sambin 1987). Hence, the Scott compactification gives a domain representation of the topological space $Pt(\mathcal{X})$, giving a connection between the two approaches of representing topological spaces (E. Palmgren, 2007).
Formal topology includes a constructive theory of domains, in the sense that the formal space $Pt(\mathcal{X})$ of a Scott (or unary) formal topology $\mathcal{X}$ is a Scott domain when ordered by inclusion, and every Scott domain arises in this way (G. Sambin, 1987).

In fact, the formal space $Pt(\mathcal{X})$ of $\mathcal{X}$ can be embedded in a Scott domain $D = (Pt(\mathcal{X}_S), \subseteq)$, where $\mathcal{X}_S$ is the Scott compactification of $\mathcal{X}$ (so $\mathcal{X}_S$ is Scott) (G. Sambin 1987). Hence, the Scott compactification gives a domain representation of the topological space $Pt(\mathcal{X})$, giving a connection between the two approaches of representing topological spaces (E. Palmgren, 2007).

This representation comes with an almost automatic lifting of morphisms: If $F : \mathcal{X} \to \mathcal{Y}$ continuous, then $F_S = \text{def} F : \mathcal{X}_S \to \mathcal{Y}_S$ is continuous and the induced continuous function $Pt(F_S) : Pt(\mathcal{X}_S) \to Pt(\mathcal{Y}_S)$ satisfies $Pt(F_S) |_{Pt(\mathcal{X})} = Pt(F)$ (E. Palmgren, 2007).
If we consider the formal reals $Pt(\mathcal{R})$, the space $Pt(\mathcal{R}_S)$ gives us a class of generalized reals, and the function lifting result lets us apply the lifted functions on these generalized reals. However, the extension $Pt(\mathcal{R}_S)$ is in some respects not so nice.
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There is another result saying that every stable locally Scott formal topology $\mathcal{X}$ give rise to a continuous dcpo $(Pt(\mathcal{X}), \subseteq)$, and that every continuous dcpo arises in this way (S. Negri 2002). There is a nice extension of the reals in this class of formal topologies, called the partial reals in (S. Negri, 2002).
If we consider the formal reals $Pt(\mathcal{R})$, the space $Pt(\mathcal{R}_S)$ gives us a class of generalized reals, and the function lifting result lets us apply the lifted functions on these generalized reals. However, the extension $Pt(\mathcal{R}_S)$ is in some respects not so nice.

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The aim now is to describe a continuous domain representation of formal topologies as continuous dcpo's, in such a way that the formal reals are represented in the partial reals. To make this work we have to restrict the class of formal topologies.
Formal Topology

Definition

A Formal Topology (with positivity predicate) is a tuple \( \mathcal{X} = (X, \leq, \triangleleft, \text{Pos}) \) where \((X, \leq)\) is a pre-ordered set (the set of basic opens) and \(\triangleleft\) is a relation between basic opens \(a \in X\) and subsets \(U \subseteq X\) satisfying

(Ref) If \(a \in U\), then \(a \triangleleft U\),

(Tra) If \(a \triangleleft U\) and \(U \triangleleft V\), then \(a \triangleleft V\),

(Ext) If \(a \leq b\), then \(a \triangleleft \{b\}\),

(Loc) If \(a \triangleleft U\) and \(a \triangleleft V\), then \(a \triangleleft U_{\leq} \cap V_{\leq}\).
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(Loc) If $a \triangleleft U$ and $a \triangleleft V$, then $a \triangleleft U \leq \cap V \leq$.

Here $U \triangleleft V$ just means that $u \triangleleft V$ for all $u \in U$ and $U \leq \triangleq \text{def} \{x \in X : (\exists u \in U) x \leq u\}$.

If $a \triangleleft U$ we say that $U$ covers $a$, and we call $\triangleleft$ a cover relation.
Formal Topology

Furthermore, Pos is a predicate on $X$ satisfying

(Mon) If $\text{Pos}(a)$ and $a \triangleleft U$, then there is $b \in U$ such that $\text{Pos}(b)$,

(Pos) For all $a \in X$, $a \triangleleft \{a\}^+$,

where $U^+ = \text{def} \{x \in X : x \in U \& \text{Pos}(x)\}$. 
Definition
A (formal) point in a formal topology $\mathcal{X}$ is a subset $\alpha \subseteq X$ satisfying

(i) there is $a \in \alpha$,

(ii) $a, b \in \alpha$ iff there is $c \in \alpha$ such that $c \leq a, b$,

(iii) if $a \in \alpha$ and $a \triangleleft U$, then there is $b \in U \cap \alpha$,

(iv) if $a \in \alpha$ then $\text{Pos}(a)$.

The clause (iv) can actually be derived using (iii) and (Pos) ($a \triangleleft a^+$ for all $a \in X$), but we will need it later.
The collection of points in a formal topology $\mathcal{X}$ is denoted $Pt(\mathcal{X})$, and the collection $\text{ext}_\mathcal{X}(U)$ of points associated with a set $U$ of basic opens is given by

$$\text{ext}_\mathcal{X}(U) = \text{def} \{ \alpha \in Pt(\mathcal{X}) : (\exists a \in U) a \in \alpha \}.$$  

The collections $\text{ext}_\mathcal{X}(U)$, $U \subseteq X$, form an ordinary topology on $Pt(\mathcal{X})$. 

Formal Topology

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**Theorem**

*If $\mathcal{X}$ is a formal topology, then $(Pt(\mathcal{X}), \subseteq)$ is a directed complete partial order (dcpo).*
Definition
We say that a pre-order \((X, \leq_X)\) is a \textit{consistently complete lower semi-lattice pre-order} (CLSP), if \(X\) has a largest element \(\top\) and if for every consistent pair \(a, b \in X\), i.e. which has a lower bound \(c \leq a\) and \(c \leq b\), there is an element \(a \wedge b \in X\) satisfying

\((a)\) \(a \wedge b \leq a, b,\)

\((b)\) \(c \leq a, b\) implies \(c \leq a \wedge b.\)

Then \(\mathcal{X} = (X, \leq, \triangleleft, \text{Pos})\) is called a CLSP formal topology whenever \((X, \leq)\) is a CLSP.
Scott compactification

Given a formal topology $\mathcal{X} = (X, \leq, \triangleleft, \text{Pos})$ we define a new structure $\mathcal{X}_S$, as follows: First we define a new preorder $\leq_S$ on $X$ by

$$a \leq_S b \iff \text{def } a \triangleleft b,$$

then we define a new relation

$$a \triangleleft_S U \iff \text{def } \text{Pos}(a) \rightarrow \exists b \in U : a \leq_S b,$$

and define $\mathcal{X}_S = (X, \leq_S, \triangleleft_S, \text{Pos})$. One can show that $\mathcal{X}_S$ is a formal topology, and we call it the *Scott compactification* of $\mathcal{X}$ (S. Negri, 2002).
Scott compactification

If $\mathcal{X}$ is a formal topology, such that $\mathcal{X}_S$ is a CLSP formal topology, then

- $(Pt(\mathcal{X}_S), \subseteq)$ is a Scott domain,
- $Pt(\mathcal{X}) \subseteq Pt(\mathcal{X}_S)$ and
- the point topology on $Pt(\mathcal{X}_S)$ is the Scott Topology.
Scott compactification

If $\mathcal{X}$ is a formal topology, such that $\mathcal{X}_S$ is a CLSP formal topology, then

- $(Pt(\mathcal{X}_S), \subseteq)$ is a Scott domain,
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- the point topology on $Pt(\mathcal{X}_S)$ is the Scott Topology.

This can then be stated in terms of a (homeomorphic) domain representation $(D, D^R, \varphi)$, with

- domain $D = (Pt(\mathcal{X}_S), \subseteq)$,
- representing elements $D^R = Pt(\mathcal{X})$ and
- representation map $\varphi = id_{Pt(\mathcal{X})} : D^R \to Pt(\mathcal{X})$. 
Given a formal topology $\mathcal{X}$ we can define a relation $\prec$ on $X$ by

$$a \prec b \iff X \prec a^\perp \cup \{b\},$$

where $a^\perp = \{c \in X : c \leq \cap a \leq \triangleleft \emptyset\}$ (the open complement of $a$).

If $a \prec b$ we say that $a$ is well covered by $b$. 
Regular Formal Topologies

Definition
A formal topology $\mathcal{X}$ is said to be regular if

$$b \triangleleft \{ x \in X : x \prec b \},$$

for all $b \in X$.

Moreover, we will say that $\prec$ is dense on $X$ if

$$a \prec b \implies (\exists c \in X) a \prec c \prec b.$$
Given a regular CLSP formal topology \( \mathcal{X} \) with \( \prec \) dense on \( X \), we can define a new topology \( \mathcal{X}_R \) as follows: Let

\[
a \leq_R b \iff \text{def } a \leq_S b,
\]

i.e. \( a \leq_R b \) iff \( a \triangleleft b \), and

\[
a \triangleleft_R U \iff \text{def } \{ c \in X : c \prec a \} \triangleleft_S U,
\]

i.e. \( a \triangleleft_R U \iff (\forall c \prec a)(\text{Pos}(c) \rightarrow (\exists b \in U)(c \triangleleft b)).\)

We set \( \mathcal{X}_R = (X, \leq_R, \triangleleft_R, \text{Pos}). \)
Proposition

If $\mathcal{X}$ is regular, CLSP and $\prec$ is dense on $X$, then $\mathcal{X}_R$ is a formal topology.

The topology $\mathcal{X}_R$ is not regular but it (trivially) satisfies

$$b \triangleleft_R \{a \in X \mid a \prec b\}$$

for all $b \in X$.

Note: the relation $\prec$ is still defined via the original cover $\triangledown$. 

Regular Formal Topologies
Lemma

The points of $\mathcal{X}_R$ are precisely those subsets $\alpha \subseteq X$ satisfying

(i) There is $a \in \alpha$,

(ii) $a, b \in \alpha$ iff there is $c \in \alpha$ such that $c \leq_R a, b$,

(iii) If $a \in \alpha$, then there is $c \in \alpha$ with $c \prec a$,

(iv) $\text{Pos}(a)$ for all $a \in \alpha$.

For any $a \in X$ with $\text{Pos}(a)$ we have

$$\uparrow a = \text{def} \{x \in X : a \prec x\} \in Pt(\mathcal{X}_R)$$
Theorem
If $\mathcal{X}$ is regular, CLSP and $\prec$ is dense on $X$, then

(a) $Pt(\mathcal{X}_R)$ is a continuous dcpo with a base given by elements $\uparrow a$ with $\text{Pos}(a)$,
(b) $Pt(\mathcal{X}) \subseteq Pt(\mathcal{X}_R)$,
(c) $Pt(\mathcal{X}) \cap \text{ext}_{\mathcal{X}_R}(U) = \text{ext}_{\mathcal{X}}(U)$.

The way below relation $\ll$ on $Pt(\mathcal{X}_R)$ has a simple characterization:

$$\alpha \ll \beta \iff (\exists b \in \beta)(\alpha \subseteq \uparrow b \subseteq \beta).$$
The point topology on $Pt(X_R)$ is precisely the Scott topology, so we can again state the above result as a domain representation $(D, D^R, \varphi)$, where

- $D = Pt(X_R)$,
- $D^R = Pt(X)$ and
- $\varphi = id_{Pt(X)} : D^R \rightarrow Pt(X)$.

As in the case of the Scott compactification, we get a lifting of continuous morphism (although not as automatic).
**Continuous Morphisms**

**Definition**

A *continuous morphism* $F : \mathcal{X} \to \mathcal{Y}$ between formal topologies is a relation $F \subseteq X \times Y$ satisfying

1. **(A1)** $aFb, b \preceq_Y V \implies a \preceq_X F^{-1}V$,
2. **(A2)** $a \preceq_X U, xFb$ for all $x \in U \implies aFb$
3. **(A3)** $X \preceq_X F^{-1}Y$,
4. **(A4)** $aFb, aFc \implies a \preceq_X F^{-1}(b \preceq_Y \cap c \preceq_Y)$

Here $F^{-1}Z = \text{def} \{ x \in X : (\exists z \in Z)xFz \}$. 
Each continuous morphism $F : \mathcal{X} \to \mathcal{Y}$ induces a continuous point function $Pt(F) : Pt(\mathcal{X}) \to Pt(\mathcal{Y})$ defined by

$$Pt(F)(\alpha) = \{ b \in \mathcal{Y} \mid \exists a \in \alpha : aFb \}.$$ 

A continuous morphism $F$ then satisfies

$$aFb \implies Pt(F)(\text{ext}_\mathcal{X}(a)) \subseteq \text{ext}_\mathcal{Y}(b),$$

for all $a \in \mathcal{X}, b \in \mathcal{Y}$. 
Continuous Morphisms

Moreover, point functions are monotone

\[ \alpha \subseteq \beta \implies \text{Pt}(F)(\alpha) \subseteq \text{Pt}(F)(\beta), \]

and preserve directed suprema

\[ \alpha_i \ (i \in I) \ \text{directed} \implies \text{Pt}(F)(\bigcup_{i \in I} \alpha_i) = \bigcup_{i \in I} \text{Pt}(F)(\alpha_i). \]

So \( \text{Pt}(F) \) is Scott continuous between dcpo’s

\( (\text{Pt}(X), \subseteq) \to (\text{Pt}(Y), \subseteq). \)
Every continuous morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ between regular CLSP formal topologies, with dense well covered relations, induces a new relation $F_R \subseteq X \times Y$ defined by

$$aF_R b \iff_{\text{def}} \{ x \in X : x \prec a \}^+ \subseteq F^{-1}\{ y \in Y : y \prec b \}.$$
Lifting Continuous Morphisms

Every continuous morphism $F : \mathcal{X} \to \mathcal{Y}$ between regular CLSP formal topologies, with dense well covered relations, induces a new relation $F_R \subseteq X \times Y$ defined by

$$aF_R b \iff_{def} \{x \in X : x \prec a\}^+ \subseteq F^{-1}\{y \in Y : y \prec b\}.$$

**Theorem**

The relation $F_R$ is a continuous morphism between $\mathcal{X}_R$ and $\mathcal{Y}_R$, and hence $Pt(F_R) : Pt(\mathcal{X}_R) \to Pt(\mathcal{Y}_R)$ is Scott continuous. Moreover, if $\alpha \in Pt(\mathcal{X}) \subseteq Pt(\mathcal{X}_R)$ then

$$Pt(F_R)(\alpha) = Pt(F)(\alpha).$$
Formal Reals $\mathcal{R}$

The basic opens of $\mathcal{R}$ are all pairs

$$(p, q) \in (\mathbb{Q} \cup \{-\infty\}) \times (\mathbb{Q} \cup \{+\infty\}),$$

where $p < q$. The pre-order is given by inclusion (as intervals), i.e.

$$(p, q) \leq (r, s) \iff r \leq p < q \leq s.$$ 

Then $\mathcal{R}$ is a CLSP formal topology with top element $(-\infty, +\infty)$ and $\land$ given by interval intersection

$$(p, q) \land (r, s) = \text{def} (\min(p, r), \max(q, s)).$$
The cover $\triangleleft$ of $\mathcal{R}$ is generated by

\[(G1)\ (p, q) \triangleleft \{(p', q') : p < p' < q' < q\}, \text{ all } (p, q),\]

\[(G2)\ (p, q) \triangleleft \{(p, s), (r, q)\}, \text{ all } (p, q) \text{ and } p \leq r < s \leq q.\]

The positivity predicate Pos is trivial since we only consider $(p, q)$ with $p < q$.

The space $Pt(\mathcal{R})$ is homeomorphic to the ordinary real numbers (S. Negri, 1996).
The well covered relation $\prec$ on basic opens is given by

$$(p, q) \prec (r, s) \iff (r < p \lor r = -\infty) \land (q < s \lor s = +\infty).$$

Hence, $\mathcal{R}$ is a regular CLSP formal topology and $\prec$ is dense.

Applying the previous results we get an extension $Pt(\mathcal{R}_R) \supseteq Pt(\mathcal{R})$ which is a continuous dcpo when ordered by inclusion.

What are the points $Pt(\mathcal{R}_R)$ of $\mathcal{R}_R$?
The dcpo $Pt(\mathcal{R}_R)$

One can show that $Pt(\mathcal{R}_R)$ is (essentially) the structure called \textit{the partial reals} in (S. Negri, 2002). These are the points of the formal topology $\mathcal{R}_p$ you get by removing (G2) from the generation of the cover of $\mathcal{R}$.
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Another description is given by the generalized reals in (F. Richman, 1998). Richman’s generalized reals are given as pairs, $(L, U)$, of disjoint open subsets, $L, U \subseteq \mathbb{Q}$ satisfying $p < q \in L \implies p \in L$ and $p > q \in U \implies p \in U$. These are ordered by inclusion:

$$(L, U) \leq (L', U') \iff \text{def } L \subseteq L' \text{ and } U' \subseteq U.$$
The dcpo $Pt(\mathcal{R}_R)$

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$$(L, U) \leq (L', U') \iff \text{def } L \subseteq L' \& U' \subseteq U.$$  

There is a bijective correspondence between such cuts and elements of $Pt(\mathcal{R}_R)$. The above ordering of cuts induces a partial order on $Pt(\mathcal{R}_R)$ that coincides with usual partial order of formal reals.
The dcpo $Pt(\mathcal{R}_R)$

Some examples of points:

$$-\infty =_{\text{def}} \{(-\infty, q) : q \in \mathbb{Q}\} \text{ and } +\infty =_{\text{def}} \{(p, +\infty) : p \in \mathbb{Q}\}.$$
The dcpo $Pt(\mathcal{R}_R)$

Some examples of points:

$-\infty = \text{def} \{(\text{-}\infty, q) : q \in \mathbb{Q}\}$ and $+\infty = \text{def} \{(p, +\infty) : p \in \mathbb{Q}\}$.

If we for every $\alpha \leq \beta \in Pt(\mathcal{R})$ define

$I[\alpha, \beta] = \text{def} \{(p, q) : (p, +\infty) \in \alpha \& (\text{-}\infty, q) \in \beta\}$,

then $I[\alpha, \beta] \in Pt(\mathcal{R}_R)$.

Points $I[p, q]$ with $p < q \in \mathbb{Q} \cup \{-\infty, +\infty\}$ make up the base of $Pt(\mathcal{R}_R)$. 
The dcpo $Pt(\mathcal{R}_R)$

Some examples of points:

$-\infty = \text{def} \{(\neg\infty, q) : q \in \mathbb{Q}\} \text{ and } +\infty = \text{def} \{(p, +\infty) : p \in \mathbb{Q}\}$.

If we for every $\alpha \leq \beta \in Pt(\mathcal{R})$ define

$I[\alpha, \beta] = \text{def} \{(p, q) : (p, +\infty) \in \alpha \text{ and } (-\infty, q) \in \beta\}$,

then $I[\alpha, \beta] \in Pt(\mathcal{R}_R)$.

Points $I[p, q]$ with $p < q \in \mathbb{Q} \cup \{-\infty, +\infty\}$ make up the base of $Pt(\mathcal{R}_R)$.

Distances between points and sets in a metric space (F. Richman, 1998).
Continuous Functions on $Pt(\mathcal{R})$

We know that every continuous morphism $F : \mathcal{R} \to \mathcal{R}$ lift to a Scott continuous function $Pt(F_{\mathcal{R}}) : Pt(\mathcal{R}_{\mathcal{R}}) \to Pt(\mathcal{R}_{\mathcal{R}})$ satisfying $Pt(F_{\mathcal{R}})(\alpha) = Pt(F)(\alpha)$ for all $\alpha \in Pt(\mathcal{R})$. 
Continuous Functions on $Pt(R)$

We know that every continuous morphism $F : R \to R$ lift to a Scott continuous function $Pt(F_R) : Pt(R_R) \to Pt(R_R)$ satisfying $Pt(F_R)(\alpha) = Pt(F)(\alpha)$ for all $\alpha \in Pt(R)$.

Now, every continuous function $f : R \to R$ (in the sense of Bishop (Bishop, Bridges, 1985)) can be represented by a continuous morphism $A_f : R \to R$ (Palmgren, 2004). Hence we can lift every such function to the dcpo $Pt(R_R)$ and apply it to nonstandard elements.

\[ f_R(I_{[a, b]}) = I_{[i, s]} \]
Continuous Functions on $Pt(\mathcal{R})$

We know that every continuous morphism $F : \mathcal{R} \to \mathcal{R}$ lift to a Scott continuous function $Pt(F_R) : Pt(\mathcal{R}_R) \to Pt(\mathcal{R}_R)$ satisfying $Pt(F_R)(\alpha) = Pt(F)(\alpha)$ for all $\alpha \in Pt(\mathcal{R})$.

Now, every continuous function $f : \mathbb{R} \to \mathbb{R}$ (in the sense of Bishop (Bishop, Bridges, 1985)) can be represented by a continuous morphism $A_f : \mathcal{R} \to \mathcal{R}$ (Palmgren, 2004). Hence we can lift every such function to the dcpo $Pt(\mathcal{R}_R)$ and apply it to nonstandard elements.

**Theorem**

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $a \leq b \in \mathbb{R}$, then $i = \inf_{x \in [a,b]} f(x)$ and $s = \sup_{x \in [a,b]} f(x)$ exist, and

$$f_R(l [a, b]) = l [i, s].$$
Arithmetic in $Pt(\mathcal{R}_R)$

Using this one can show that arithmetic on $Pt(\mathcal{R}_R)$ coincides with interval arithmetic on interval points: for $a \leq b, c \leq d \in \mathbb{R}$, we have

$$I[a, b] +_R I[c, d] = I[a + c, b + d]$$

and

$$I[a, b] \times_R I[c, d] = I[\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)].$$
In (A. Edalat & A. Lieutier, 2001) the authors introduced an operation $D$ on functions $f : \mathbb{I}\mathbb{R} \to \mathbb{I}\mathbb{R}$, where $\mathbb{I}\mathbb{R}$ consists of all closed real intervals together with $\mathbb{R}$ ordered under reverse inclusion, satisfying

- $D(f)$ continuous for every continuous $f : \mathbb{I}\mathbb{R} \to \mathbb{I}\mathbb{R}$,
- $D(\mathbf{I}f) = \mathbf{I}\left(\frac{df}{dx}\right)$, for all $C^1$ functions $f : \mathbb{R} \to \mathbb{R}$.

Here $\mathbf{I}(f)$ is the lifting of $f$ given by the direct image under $f$. 
In our setting (i.e. replacing $\mathbb{I}\mathbb{R}$ by $Pt(\mathcal{R}_R)$) we can show that $D(f_R)$ is well defined when $f : \mathbb{R} \to \mathbb{R}$ is differentiable (again in the sense of Bishop) and

$$D(f_R) = \left( \frac{df}{dx} \right)_R.$$
Thank You!
References


