A Continuous Dcpo Representation of Regular Formal Topologies

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Background/Motivation

- Formal topology includes a constructive theory of domains, in the sense that the formal space $Pt(\mathcal{X})$ of a *Scott (or unary) formal topology* $\mathcal{X}$ is a Scott domain when ordered by inclusion, and every Scott domain arises in this way (G. Sambin, 1987).

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- In fact, the formal space $Pt(\mathcal{X})$ of $\mathcal{X}$ can be embedded in a Scott domain $D = (Pt(\mathcal{X}_S), \subseteq)$, where $\mathcal{X}_S$ is the Scott compactification of $\mathcal{X}$ (so $\mathcal{X}_S$ is Scott) (G. Sambin 1987). Hence, the Scott compactification gives a domain representation of the topological space $Pt(\mathcal{X})$, giving a connection between the two approaches of representing topological spaces (E. Palmgren, 2007).

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- This representation comes with an almost automatic lifting of morphisms: If $F : \mathcal{X} \to \mathcal{Y}$ continuous, then $F_S =_{\mathrm{def}} F : \mathcal{X}_S \to \mathcal{Y}_S$ is continuous and the induced continuous function $Pt(F_S) : Pt(\mathcal{X}_S) \to Pt(\mathcal{Y}_S)$ satisfies $Pt(F_S) \mid_{Pt(\mathcal{X})} = Pt(F)$ (E. Palmgren, 2007).

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- There is another result saying that every *stable locally Scott* formal topology $\mathcal{X}$ give rise to a continuous dcpo $(Pt(\mathcal{X}), \subseteq)$, and that every continuous dcpo arises in this way (S. Negri 2002). There is a nice extension of the reals in this class of formal topologies, called *the partial reals* in (S. Negri, 2002).

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- The aim now is to describe a continuous domain representation of formal topologies as continuous dcpo's, in such a way that the formal reals are represented in the partial reals. To make this work we have to restrict the class of formal topologies.

Formal Topology

Definition

A *Formal Topology* (with positivity predicate) is a tuple $\mathcal{X} = (X, \leq, \lhd, \mathrm{Pos})$ where $(X, \leq)$ is a pre-ordered set (the set of *basic opens*) and $\lhd$ is a relation between basic opens $a \in X$ and subsets $U \subseteq X$ satisfying

(Ref) If $a \in U$, then $a \lhd U$,

(Tra) If $a \lhd U$ and $U \lhd V$, then $a \lhd V$,

(Ext) If $a \leq b$, then $a \lhd \{b\}$,

(Loc) If $a \lhd U$ and $a \lhd V$, then $a \lhd U_{\leq} \cap V_{\leq}$.

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(Loc) If $a \lhd U$ and $a \lhd V$, then $a \lhd U_{\leq} \cap V_{\leq}$.

Here $U \lhd V$ just means that $u \lhd V$ for all $u \in U$ and $U_{\leq} =_{\mathrm{def}} \{x \in X : (\exists u \in U) x \leq u\}$.

If $a \lhd U$ we say that U covers a, and we call $\lhd$ a cover relation.

Formal Topology

Furthermore, Pos is a predicate on X satisfying

(Mon) If $\mathrm{Pos}(a)$ and $a \lhd U$, then there is $b \in U$ such that $\mathrm{Pos}(b)$,

(Pos) For all $a \in X$, $a \lhd \{a\}^+$,

where $U^+ =_{\mathrm{def}} \{x \in X : x \in U \;\&\; \mathrm{Pos}(x)\}$.

Formal Topology

Definition

A *(formal) point* in a formal topology $\mathcal{X}$ is a subset $\alpha \subseteq X$ satisfying

(i) there is $a \in \alpha$,

(ii) $a, b \in \alpha$ iff there is $c \in \alpha$ such that $c \leq a, b$,

(iii) if $a \in \alpha$ and $a \lhd U$, then there is $b \in U \cap \alpha$,

(iv) if $a \in \alpha$ then $\mathrm{Pos}(a)$.

The clause (iv) can actually be derived using (iii) and (Pos) ($a \lhd a^+$ for all $a \in X$), but we will need it later.

Formal Topology

The collection of points in a formal topology $\mathcal{X}$ is denoted $Pt(\mathcal{X})$, and the collection $\mathrm{ext}_{\mathcal{X}}(U)$ of points associated with a set U of basic opens is given by

$$\mathrm{ext}_{\mathcal{X}}(U) =_{\mathrm{def}} \{\alpha \in Pt(\mathcal{X}) : (\exists a \in U) a \in \alpha\}.$$

The collections $\mathrm{ext}_{\mathcal{X}}(U)$, $U \subseteq X$, form an ordinary topology on $Pt(\mathcal{X})$.

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Theorem

If $\mathcal{X}$ is a formal topology, then $(Pt(\mathcal{X}), \subseteq)$ is a directed complete partial order (dcpo).

CLSP Formal Topologies

Definition

We say that a pre-order $(X, \leq_{\mathcal{X}})$ is a *consistently complete lower semi-lattice pre-order* (CLSP), if X has a largest element $\top$ and if for every consistent pair $a, b \in X$, i.e. which has a lower bound $c \leq a$ and $c \leq b$, there is an element $a \wedge b \in X$ satisfying

(a) $a \wedge b \leq a, b$,

(b) $c \leq a, b$ implies $c \leq a \wedge b$.

Then $\mathcal{X} = (X, \leq, \triangleleft, \mathrm{Pos})$ is called a CLSP formal topology whenever $(X, \leq)$ is a CLSP.

Scott compactification

Given a formal topology $\mathcal{X} = (X, \leq, \lhd, \mathrm{Pos})$ we define a new structure $\mathcal{X}_S$, as follows: First we define a new preorder $\leq_S$ on X by

$$a \leq_S b \iff_{\mathrm{def}} a \lhd b,$$

then we define a new relation

$$a \lhd_S U \iff_{\mathrm{def}} \mathrm{Pos}(a) \to \exists b \in U : a \leq_S b,$$

and define $\mathcal{X}_S = (X, \leq_S, \lhd_S, \mathrm{Pos})$. One can show that $\mathcal{X}_S$ is a formal topology, and we call it the *Scott compactification* of $\mathcal{X}$ (S. Negri, 2002).

Scott compactification

If $\mathcal{X}$ is a formal topology, such that $\mathcal{X}_S$ is a CLSP formal topology, then

- $(Pt(\mathcal{X}_S), \subseteq)$ is a Scott domain,
- $Pt(\mathcal{X}) \subseteq Pt(\mathcal{X}_S)$ and
- the point topology on $Pt(\mathcal{X}_S)$ is the Scott Topology.

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This can then be stated in terms of a (homeomorphic) domain representation (D, D^R, φ), with

- domain $D = (Pt(\mathcal{X}_S), \subseteq)$,
- representing elements $D^R = Pt(\mathcal{X})$ and
- representation map $\varphi = id_{Pt(\mathcal{X})} : D^R \to Pt(\mathcal{X})$.

Regular Formal Topologies

Given a formal topology $\mathcal{X}$ we can define a relation $\prec$ on X by

$$a \prec b \Longleftrightarrow_{\mathsf{def}} X \lhd a^{\perp} \cup \{b\},$$

where $a^{\perp} =_{\mathsf{def}} \{c \in X : c_{\leq} \cap a_{\leq} \lhd \emptyset\}$ (the *open complement* of a).

If $a \prec b$ we say that a is *well covered* by b.

Regular Formal Topologies

Definition

A formal topology $\mathcal{X}$ is said to be *regular* if

$$b \lhd \{x \in X : x \prec b\},$$

for all $b \in X$.

Moreover, we will say that $\prec$ is *dense* on X if

$$a \prec b \Longrightarrow (\exists c \in X) a \prec c \prec b.$$

Regular Formal Topologies

Given a regular CLSP formal topology $\mathcal{X}$ with $\prec$ dense on X, we can define a new topology $\mathcal{X}_R$ as follows: Let

$$a \leq_R b \iff_{\text{def}} a \leq_S b,$$

i.e. $a \leq_R b$ iff $a \lhd b$, and

$$a \lhd_R U \iff_{\text{def}} \{c \in X : c \prec a\} \lhd_S U,$$

i.e. $a \lhd_R U \iff (\forall c \prec a)(\mathrm{Pos}(c) \to (\exists b \in U)(c \lhd b))$.

We set $\mathcal{X}_R = (X, \leq_R, \lhd_R, \mathrm{Pos})$.

Regular Formal Topologies

Proposition

If $\mathcal{X}$ is regular, CLSP and $\prec$ is dense on X, then $\mathcal{X}_R$ is a formal topology.

The topology $\mathcal{X}_R$ is not regular but it (trivially) satisfies

$$b \lhd_R \{a \in X \mid a \prec b\}$$

for all $b \in X$.
Note: the relation $\prec$ is still defined via the original cover $\lhd$.

Points of $\mathcal{X}_R$

Lemma

The points of $\mathcal{X}_R$ are precisely those subsets $\alpha \subseteq X$ satisfying

(i) *There is $a \in \alpha$,*

(ii) *$a, b \in \alpha$ iff there is $c \in \alpha$ such that $c \leq_R a, b$,*

(iii) *If $a \in \alpha$, then there is $c \in \alpha$ with $c \prec a$,*

(iv) *$Pos(a)$ for all $a \in \alpha$.*

For any $a \in X$ with $\mathrm{Pos}(a)$ we have

$$\uparrow a =_{\mathrm{def}} \{x \in X : a \prec x\} \in Pt(\mathcal{X}_R)$$

Regular Formal Topologies and Continuous Dcpo's

Theorem

If $\mathcal{X}$ is regular, CLSP and $\prec$ is dense on X, then

(a) $Pt(\mathcal{X}_R)$ *is a continuous dcpo with a base given by elements* $\uparrow a$ *with* $Pos(a)$,

(b) $Pt(\mathcal{X}) \subseteq Pt(\mathcal{X}_R)$,

(c) $Pt(\mathcal{X}) \cap ext_{\mathcal{X}_R}(U) = ext_{\mathcal{X}}(U)$.

The way below relation $\ll$ on $Pt(\mathcal{X}_R)$ has a simple characterization:

$$\alpha \ll \beta \Longleftrightarrow (\exists b \in \beta)(\alpha \subseteq \uparrow b \subseteq \beta).$$

Regular Formal Topologies and Continuous Dcpo's

The point topology on $Pt(\mathcal{X}_R)$ is precisely the Scott topology, so we can again state the above result as a domain representation (D, D^R, φ), where

- $D = Pt(\mathcal{X}_R)$,
- $D^R = Pt(\mathcal{X})$ and
- $\varphi = id_{Pt(\mathcal{X})} : D^R \to Pt(\mathcal{X})$.

As in the case of the Scott compactification, we get a lifting of continuous morphism (although not as automatic).

Continuous Morphisms

Definition

A *continuous morphism* $F : \mathcal{X} \to \mathcal{Y}$ between formal topologies is a relation $F \subseteq X \times Y$ satisfying

(A1) $aFb,\ b \lhd_{\mathcal{Y}} V \Longrightarrow a \lhd_{\mathcal{X}} F^{-1}V,$

(A2) $a \lhd_{\mathcal{X}} U,\ xFb$ for all $x \in U \Longrightarrow aFb$

(A3) $X \lhd_{\mathcal{X}} F^{-1}Y,$

(A4) $aFb,\ aFc \Longrightarrow a \lhd_{\mathcal{X}} F^{-1}(b_{\leq_{\mathcal{Y}}} \cap c_{\leq_{\mathcal{Y}}})$

Here $F^{-1}Z =_{\text{def}} \{x \in X : (\exists z \in Z) xFz\}$.

Continuous Morphisms

Each continuous morphism $F : \mathcal{X} \to \mathcal{Y}$ induces a continuous point function $Pt(F) : Pt(\mathcal{X}) \to Pt(\mathcal{Y})$ defined by

$$Pt(F)(\alpha) = \{b \in Y \mid \exists a \in \alpha : aFb\}.$$

A continuous morphism F then satisfies

$$aFb \Longrightarrow Pt(F)(\mathrm{ext}_{\mathcal{X}}(a)) \subseteq \mathrm{ext}_{\mathcal{Y}}(b),$$

for all $a \in X$, $b \in Y$.

Continuous Morphisms

Moreover, point functions are monotone

$$\alpha \subseteq \beta \Longrightarrow Pt(F)(\alpha) \subseteq Pt(F)(\beta),$$

and preserve directed suprema

$$\alpha_i \ (i \in I) \text{ directed } \Longrightarrow Pt(F)(\bigcup_{i \in I} \alpha_i) = \bigcup_{i \in I} Pt(F)(\alpha_i).$$

So $Pt(F)$ is Scott continuous between dcpo's $(Pt(\mathcal{X}), \subseteq) \to (Pt(\mathcal{Y}), \subseteq)$.

Lifting Continuous Morphisms

Every continuous morphism $F : \mathcal{X} \to \mathcal{Y}$ between regular CLSP formal topologies, with dense well covered relations, induces a new relation $F_R \subseteq X \times Y$ defined by

$$aF_R b \iff_{\text{def}} \{x \in X : x \prec a\}^+ \subseteq F^{-1}\{y \in Y : y \prec b\}.$$

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Theorem

The relation F_R is a continuous morphism between $\mathcal{X}_R$ and $\mathcal{Y}_R$, and hence $Pt(F_R) : Pt(\mathcal{X}_R) \to Pt(\mathcal{Y}_R)$ is Scott continuous. Moreover, if $\alpha \in Pt(\mathcal{X}) \subseteq Pt(\mathcal{X}_R)$ then

$$Pt(F_R)(\alpha) = Pt(F)(\alpha).$$

Formal Reals $\mathcal{R}$

The basic opens of $\mathcal{R}$ are all pairs

$$(p, q) \in (\mathbb{Q} \cup \{-\infty\}) \times (\mathbb{Q} \cup \{+\infty\}),$$

where $p < q$. The pre-order is given by inclusion (as intervals), i.e.

$$(p, q) \leq (r, s) \Longleftrightarrow_{\mathrm{def}} r \leq p < q \leq s.$$

Then $\mathcal{R}$ is a CLSP formal topology with top element $(-\infty, +\infty)$ and $\wedge$ given by interval intersection

$$(p, q) \wedge (r, s) =_{\mathrm{def}} (\min(p, r), \max(q, s)).$$

Formal Reals $\mathcal{R}$

The cover $\triangleleft$ of $\mathcal{R}$ is generated by

(G1) $(p, q) \triangleleft \{(p', q') : p < p' < q' < q\}$, all (p, q),

(G2) $(p, q) \triangleleft \{(p, s), (r, q)\}$, all (p, q) and $p \leq r < s \leq q$.

The positivity predicate Pos is trivial since we only consider (p, q) with $p < q$.

The space $Pt(\mathcal{R})$ is homeomorphic to the ordinary real numbers (S. Negri, 1996).

Formal Reals $\mathcal{R}$

The well covered relation $\prec$ on basic opens is given by

$$(p, q) \prec (r, s) \iff (r < p \vee r = -\infty) \ \& \ (q < s \vee s = +\infty).$$

Hence, $\mathcal{R}$ is a regular CLSP formal topology and $\prec$ is dense.

Applying the previous results we get an extension $Pt(\mathcal{R}_R) \supseteq Pt(\mathcal{R})$ which is a continuous dcpo when ordered by inclusion.

What are the points $Pt(\mathcal{R}_R)$ of $\mathcal{R}_R$?

The dcpo $Pt(\mathcal{R}_R)$

One can show that $Pt(\mathcal{R}_R)$ is (essentially) the structure called *the partial reals* in (S. Negri, 2002). These are the points of the formal topology $\mathcal{R}_p$ you get by removing (G2) from the generation of the cover of $\mathcal{R}$.

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Another description is given by the generalized reals in (F. Richman, 1998). Richman's generalized reals are given as pairs, (L, U), of disjoint open subsets, $L, U \subseteq \mathbb{Q}$ satisfying $p < q \in L \implies p \in L$ and $p > q \in U \implies p \in U$. These are ordered by inclusion:

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$$(L, U) \leq (L', U') \Longleftrightarrow_{\mathrm{def}} L \subseteq L' \ \& \ U' \subseteq U.$$

There is a bijective correspondence between such cuts and elements of $Pt(\mathcal{R}_R)$. The above ordering of cuts induces a partial order on $Pt(\mathcal{R}_R)$ that coincides with usual partial order of formal reals.

The dcpo $Pt(\mathcal{R}_R)$

Some examples of points:

$$-\infty =_{\mathsf{def}} \{(-\infty, q) : q \in \mathbb{Q}\} \text{ and } +\infty =_{\mathsf{def}} \{(p, +\infty) : p \in \mathbb{Q}\}.$$

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If we for every $\alpha \leq \beta \in Pt(\mathcal{R})$ define

$$I[\alpha, \beta] =_{\mathrm{def}} \{(p, q) : (p, +\infty) \in \alpha \;\&\; (-\infty, q) \in \beta\},$$

then $I[\alpha, \beta] \in Pt(\mathcal{R}_R)$.

Points $I[p, q]$ with $p < q \in \mathbb{Q} \cup \{-\infty, +\infty\}$ make up the base of $Pt(\mathcal{R}_R)$.

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Distances between points and sets in a metric space (F. Richman, 1998).

Continuous Functions on $Pt(\mathcal{R})$

We know that every continuous morphism $F : \mathcal{R} \to \mathcal{R}$ lift to a Scott continuous function $Pt(F_R) : Pt(\mathcal{R}_R) \to Pt(\mathcal{R}_R)$ satisfying $Pt(F_R)(\alpha) = Pt(F)(\alpha)$ for all $\alpha \in Pt(\mathcal{R})$.

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Now, every continuous function $f : \mathbb{R} \to \mathbb{R}$ (in the sense of Bishop (Bishop, Bridges, 1985)) can be represented by a continuous morphism $A_f : \mathcal{R} \to \mathcal{R}$ (Palmgren, 2004). Hence we can lift every such function to the dcpo $Pt(\mathcal{R}_R)$ and apply it to nonstandard elements.

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Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $a \leq b \in \mathbb{R}$, then $i = \inf_{x \in [a,b]} f(x)$ and $s = \sup_{x \in [a,b]} f(x)$ exist, and

$$f_R(I\,[a,b]) = I\,[i,s]\,.$$

Arithmetic in $Pt(\mathcal{R}_R)$

Using this one can show that arithmetic on $Pt(\mathcal{R}_R)$ coincides with interval arithmetic on interval points: for $a \leq b, c \leq d \in \mathbb{R}$, we have

$$I[a,b] +_R I[c,d] = I[a+c, b+d]$$

and

$$I[a,b] \times_R I[c,d] = I[\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)].$$

Domain Theoretic Derivative

In (A. Edalat & A. Lieutier, 2001) the authors introduced an operation D on functions $f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$, where $\mathbf{I}\mathbb{R}$ consists of all closed real intervals together with $\mathbb{R}$ ordered under reverse inclusion, satisfying

- $D(f)$ continuous for every continuous $f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$,
- $D(\mathbf{I}f) = \mathbf{I}(\frac{df}{dx})$, for all $\mathcal{C}^1$ functions $f : \mathbb{R} \to \mathbb{R}$.

Here $\mathbf{I}(f)$ is the lifting of f given by the direct image under f.

Domain Theoretic Derivative

In our setting (i.e. replacing $\mathbf{I}\mathbb{R}$ by $Pt(\mathcal{R}_R)$) we can show that $D(f_R)$ is well defined when $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable (again in the sense of Bishop) and

$$D(f_R) = \left(\frac{df}{dx}\right)_R.$$

Thank You!

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