Similarity, Topology, and Uniformity

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Distance vs. Similarity

- If x moves towards y,
  - the distance between x and y gets smaller
  - the similarity of x and y gets larger
- Similarity is dual to distance:

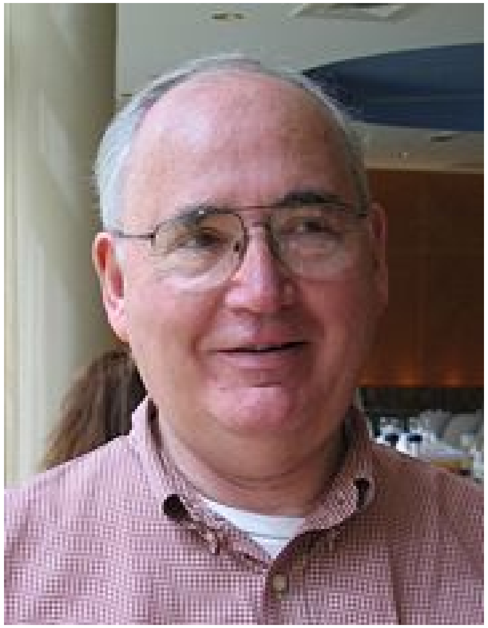
$\delta(x,z) \leq \delta(x,y) + \delta(y,z)$	$\sigma(x,y) * \sigma(y,z) \leq \sigma(x,z)$
$\delta(x,x') < r$	$\sigma(x,x') > r$
$\delta(x,x') \in U$ co-Scott open	$\sigma(x,x') \in U$ Scott open

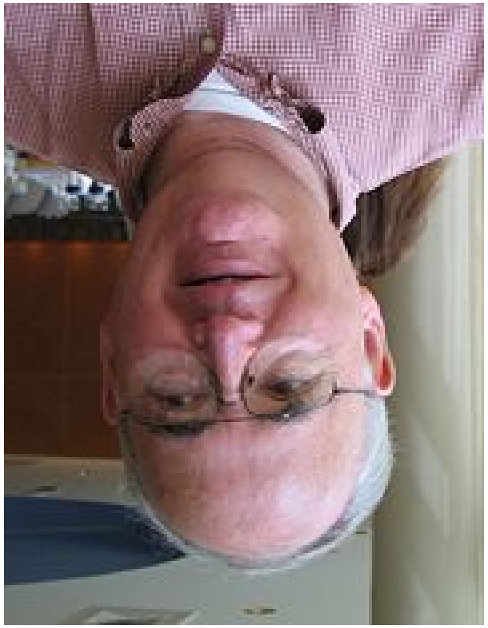
- ...but these are essentially two views of the same thing.

Similarity Preferred

Here, I prefer the similarity view

(Scott better than co-Scott).





Generalized Similarity Systems

- (X, S, σ) with
  - X set of points
  - S topological space (T_0)
  - $\sigma : X \times X \to S$ similarity function
- No axioms for the beginning
- Similarity system:

  S is restricted to be a continuous lattice with its Scott topology

Induced Neighborhood Spaces

- $\sigma : X \times X \to S$ similarity function
- Right "pre-open" ball: $\mathrm{B}^{\mathrm{R}}(x,u) = \{x' \in X \mid \sigma(x,x') \in u\}$
  for $x \in X$ and u open in S.
- $A \subseteq X$ is right neighborhood of $x \in X$
  if there is u open in S such that $x \in \mathrm{B}^{\mathrm{R}}(x,u) \subseteq A$
- Right neighborhood space $\mathrm{N}^{\mathrm{R}}(X,S,\sigma)$
- Analogously:

  Left "pre-open" ball: $\mathrm{B}^{\mathrm{L}}(x,u) = \{x' \in X \mid \sigma(x',x) \in u\}$

  Left neighborhood space $\mathrm{N}^{\mathrm{L}}(X,S,\sigma)$

Open Balls and Triangle Inequality

- $A \subseteq X$ is right open
  if A is right neighborhood of all its elements.
- Are the right “pre-open” balls right open?
- In (quasi/partial) metric spaces,
  this is derived from the triangle inequality.
- How much can this be generalized?
- Quite far: The “addition” in the triangle inequality
  need not be commutative, associative, . . . ,
  and can be parameterized on the “mid point”.

Local and Global Transitivity

- (X, S, σ) is locally transitive
  if there is a family $(*_y)_{y \in X}$ of functions
  $*_y : S \times S \to S$ continuous in each argument separately
  such that
  - $\sigma(x,y) *_y \sigma(y,z) \leq \sigma(x,z)$
  - $\sigma(x,y) *_y \sigma(y,y) = \sigma(x,y)$
  - $\sigma(y,y) *_y \sigma(y,z) = \sigma(y,z)$
- (X, S, σ) is globally transitive
  if it is locally transitive in a way such that
  all the operations $*_y$ for $y \in X$ are identical.

Consequences of Local Transitivity

- All left/right pre-open balls are left/right open.
- Hence $\mathrm{N}^{\mathrm{L}}\mathcal{X}$ and $\mathrm{N}^{\mathrm{R}}\mathcal{X}$ are topological spaces ($\mathrm{T}^{\mathrm{L}}\mathcal{X}$ and $\mathrm{T}^{\mathrm{R}}\mathcal{X}$).
- $\sigma_{\mathcal{X}} : \mathrm{T}^{\mathrm{L}}\mathcal{X} \times \mathrm{T}^{\mathrm{R}}\mathcal{X} \to S_{\mathcal{X}}$ is separately continuous.
- If all $*_y : S_{\mathcal{X}} \times S_{\mathcal{X}} \to S_{\mathcal{X}}$ are jointly continuous, then so is $\sigma_{\mathcal{X}}$.
- Remark: If $S_{\mathcal{X}}$ is a continuous lattice, separate continuity and joint continuity of $*_y$ are the same.

Comparison with Metric Notions

- GlTr1: $\sigma(x,y) * \sigma(y,z) \leq \sigma(x,z)$
  GlTr2: $\sigma(x,x) * \sigma(x,y) = \sigma(x,y) = \sigma(x,y) * \sigma(y,y)$
- Special case $(S, *, \sigma) = (\mathbb{R}^{+\mathrm{op}}, +, \delta)$:
  GlTr1: $\delta(x,z) \leq_{\mathbb{R}} \delta(x,y) + \delta(y,z)$
  GlTr2: $\delta(x,x) + \delta(x,y) = \delta(x,y) \Leftrightarrow \delta(x,x) = 0$
  Pseudo quasi metric
- Special case $(S, *, \sigma) = (\mathbb{R}^{+\mathrm{op}}, \wedge, \delta)$:
  GlTr1: $\delta(x,z) \leq_{\mathbb{R}} \delta(x,y) \vee_{\mathbb{R}} \delta(y,z)$
  GlTr2: $\delta(x,x) \vee_{\mathbb{R}} \delta(x,y) = \delta(x,y) \Leftrightarrow \delta(x,x) \leq_{\mathbb{R}} \delta(x,y)$
  "Pseudo partial ultrametric"

On Partial Metrics

- Partial metrics in the standard sense with
  $\delta(x,z) \leq_{\mathbb{R}} \delta(x,y)+\delta(y,z)-\delta(y,y)$
  are locally transitive with $a *_y b \;=\; (a+b) \dot{-} \delta(y,y)$.

- Yet a standard example for partial metrics
  can be considered as
  a globally transitive similarity system:
  $X \;=\;$ set of finite and infinite strings
  $(S, *) \;=\; (\mathbb{N} \cup \{\infty\}, \wedge)$
  $\sigma(x,y) \;=\;$ length of longest common prefix of x and y

The Morphisms

- One could fix some S and consider functions $f : (X, S, \sigma_X) \to (Y, S, \sigma_Y)$ such that $\sigma_X(x, x') \leq \sigma_Y(fx, fx')$
- Yet every X should have its own S_X
- Idea: Take $f : (X, S_X, \sigma_X) \to (Y, S_Y, \sigma_Y)$ such that there is continuous $\varphi : S_X \to S_Y$ with $\varphi(\sigma_X(x, x')) \leq \sigma_Y(fx, fx')$
- Yet this alone is too weak; $\varphi = (a \mapsto \perp)$ would make all functions to morphisms in case of a continuous lattice S_Y

Globally and Locally Continuous

- Consider (X, S_X, σ_X), (Y, S_Y, σ_Y), and $f : X \to Y$.

- f is globally continuous (GC)
  if there is a continuous $\varphi : S_X \to S_Y$ such that
  $\varphi(\sigma_X(x,x')) \leq \sigma_Y(fx, fx')$ and $\varphi(\sigma_X(x,x)) = \sigma_Y(fx, fx)$.

- f is right locally continuous (RLC)
  if for every $x \in X$ there is a continuous $\varphi_x^{\mathrm{R}} : S_X \to S_Y$ such that
  $\varphi_x^{\mathrm{R}}(\sigma_X(x,x')) \leq \sigma_Y(fx, fx')$ and $\varphi_x^{\mathrm{R}}(\sigma_X(x,x)) = \sigma_Y(fx, fx)$.

- f is left locally continuous (LLC)
  if for every $x \in X$ there is a continuous $\varphi_x^{\mathrm{L}} : S_X \to S_Y$ such that
  $\varphi_x^{\mathrm{L}}(\sigma_X(x',x)) \leq \sigma_Y(fx', fx)$ and $\varphi_x^{\mathrm{L}}(\sigma_X(x,x)) = \sigma_Y(fx, fx)$.

- f is locally continuous (LC) if it is RLC and LLC.

Characterization of Local Continuity

- For generalized similarity spaces $\mathcal{X}$ and $\mathcal{Y}$:
  - $f : \mathcal{X} \to \mathcal{Y}$ RLC $\Rightarrow f : \mathrm{N}^{\mathrm{R}}\mathcal{X} \to \mathrm{N}^{\mathrm{R}}\mathcal{Y}$ continuous
  - $f : \mathcal{X} \to \mathcal{Y}$ LLC $\Rightarrow f : \mathrm{N}^{\mathrm{L}}\mathcal{X} \to \mathrm{N}^{\mathrm{L}}\mathcal{Y}$ continuous
  - $f : \mathcal{X} \to \mathcal{Y}$ LC $\Rightarrow f$ continuous w.r.t. N^{R} and N^{L}
- For similarity spaces $\mathcal{X}$ and $\mathcal{Y}$
  (i.e. $S_{\mathcal{X}}$ and $S_{\mathcal{Y}}$ are continuous lattices),
  these implications are equivalences.

Categorical Equivalences with LC Functions

- For every bitopological space $(X, \tau^{\mathrm{L}}, \tau^{\mathrm{R}})$, there is a globally transitive similarity space $\mathcal{X}$ such that $\mathrm{T}^{\mathrm{L}}\mathcal{X} = (X, \tau^{\mathrm{L}})$ and $\mathrm{T}^{\mathrm{R}}\mathcal{X} = (X, \tau^{\mathrm{R}})$.
- If $\tau^{\mathrm{L}} = \tau^{\mathrm{R}}$, $\mathcal{X}$ can be chosen to be symmetric i.e. $\sigma_{\mathcal{X}}(x_1, x_2) = \sigma_{\mathcal{X}}(x_2, x_1)$.
- Locally transitive similarity spaces + LC functions $\cong$ globally transitive similarity spaces + LC functions $\cong$ bitopological spaces + pairwise continuous functions
- Symmetric loc./glob. trans. sim. spaces + LC fu. $\cong$ topological spaces + continuous functions

Characterization of Global Continuity

- For generalized similarity spaces $\mathcal{X}$ and $\mathcal{Y}$:
  $f : \mathcal{X} \to \mathcal{Y}$ GC (i.e. $\exists \varphi : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ such that ...) $\Rightarrow$
  f is uniformly continuous (UC) in the following sense:
  for every x in X
  and open v of $S_{\mathcal{Y}}$ containing $\sigma_{\mathcal{Y}}(fx, fx)$,
  there is an open u of $S_{\mathcal{X}}$ containing $\sigma_{\mathcal{X}}(x,x)$
  such that $\sigma_{\mathcal{X}}(x_1,x_2) \in u \Rightarrow \sigma_{\mathcal{Y}}(fx_1, fx_2) \in v.$

  (For metric spaces, this is equivalent to the usual UC.)

- For similarity spaces $\mathcal{X}$ and $\mathcal{Y}$
  (i.e. $S_{\mathcal{X}}$ and $S_{\mathcal{Y}}$ are continuous lattices),
  GC is equivalent to UC.

Categorical Equivalences with GC Functions

- Similarity spaces + GC functions $\cong$
  sets X with topology on X^2 + functions f such that
  f^2 is continuous on the diagonal

- Globally transitive similarity spaces
  with a constant value for $\sigma(x,x)$
  + GC functions $\cong$
  quasi-uniform spaces + uniformly continuous functions

- Symmetric ... + GC functions $\cong$
  uniform spaces + uniformly continuous functions

- ... with ω-continuous S_X $\cong$ ... with countable basis

Common Proof Pattern

- To prove these equivalences, one must construct a continuous φ such that

  $$\varphi(\sigma_X(x,x')) \leq \sigma_Y(fx,fx')$$
  $$\varphi(\sigma_X(x,x)) = \sigma_Y(fx,fx)$$

  or a continuous $*$ such that

  $$\sigma(x,y) * \sigma(y,z) \leq \sigma(x,z)$$
  $$\sigma(x,x) * \sigma(x,y) = \sigma(x,y) = \sigma(x,y) * \sigma(y,y)$$

- Common pattern:
  general inequality + equality for specific arguments

- Use generalized injectivity of continuous lattices.

Generalized Injectivity of Cont. Lattices

$$\begin{tikzcd} Y \arrow[r, "F"] & L \\ X \arrow[u, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

- X set
- (Y, OY) topological space
- (L, OL) continuous lattice with Scott top.

- Define $Fy = \bigvee\{\bigwedge f^{+}(g^{-}U) \mid U \in OY, U \ni y\}$

- $F : Y \to L$ is continuous and satisfies $F \circ g \leq f$.

- If a specific point x of X has the property that for all $V \in OL$ containing fx there is $U \in OY$ containing gx such that $g^{-}U \subseteq f^{-}V$, then $F(gx) = fx$ holds for this x.

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