On Domain Theory over Girard Quantales

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KEYWORDS

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Keywords

- A GMS (generalized metric space) is a set with a distance mapping of type $X \times X \to [0, 1]$ satisfying some of the usual metric axioms.
- We can furher generalize distance to type $X \times X \to \mathcal{Q}$, where $\mathcal{Q}$ is a Girard quantale.

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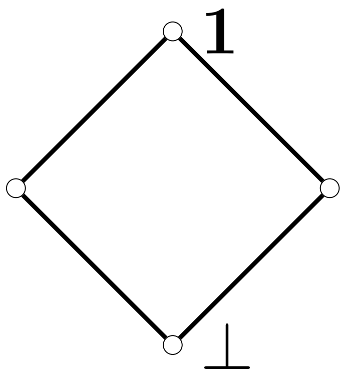
- unit: $\mathbf{1} := \neg\bot$,
- par: $a \invamp b := \neg(\neg a \otimes \neg b)$,
- Informally: $\wedge, \vee, \otimes, \invamp, \multimap, \bigvee, \bigwedge, \mathbf{1}, \bot, \neg, !, ?$.

Examples

- Every complete Boolean algebra is a Girard quantale with $\otimes = \wedge$, e.g.:



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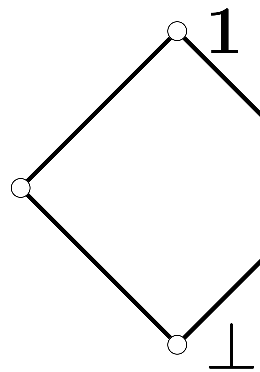

- The two-element lattice $\mathbf{2} = \{\mathbf{1}, \perp\}$ with $\otimes = \wedge$.

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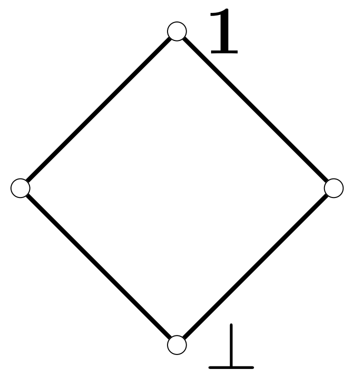

- The two-element lattice $\mathbf{2} = \{\mathbf{1}, \perp\}$ with $\otimes = \wedge$.
- The unit interval $([0, 1], \geqslant)$ with $\otimes = +$.

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MOTIVATION

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Generalized Metric Spaces

Perhaps the theory of GMSes is not as much concerned with generalizing metric spaces as with generalizing dcpos and domains:

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America, P., Rutten, J. (1989)
Solving Reflexive Domain Equations in a Category of Complete Metric Spaces, *J. Comput. Syst. Sci.* **39**(3), pp. 343–375.

Flagg, R.C., Kopperman, R. (1995)
Fixed points and reflexive domain equations in categories of continuity spaces, ENTCS **1**.

are devoted to solving recursive domain equations in GMSes.

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Rutten, J. (1996) Elements of generalized ultrametric domain theory,
Theoretical Computer Science **170**, pp. 349–381.

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Continuity Spaces: Reconciling Domains and Metric Spaces,
Theoretical Computer Science **177**(1), pp. 111–138.

Flagg, R. (1997) Quantales and continuity spaces,
Algebra Universalis **37**, pp. 257–276.

speak about generalized Alexandroff and Scott topologies.

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Bonsangue, M.M., van Breugel, F. and Rutten, J.J.M.M. (1998) Generalized Metric Spaces: Completion, Topology, and Powerdomains via the Yoneda Embedding, *Theoretical Computer Science* **193**(1-2), pp. 1–51.

proposes powerdomains for GMSes.

Generalized Metric Spaces

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- the theory is developed towards applications in denotational semantics;

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This situation is not surprising, since:

- the theory is developed towards applications in denotational semantics;
- the theorems of Scott's domain theory are universal and prone to generalizations.

On the inverse limit construction

“The pre-order version was discovered first [...]. The metric version was mainly developed by P.America and J.Rutten.

The proofs look astonishingly similar but until now the preconditions for the pre-order and the metric versions have seemed to be fundamentally different.

In this thesis we indicate how to use one and the same proof for both cases, just varying the logic to move from one setting to the other.”

(K.R. Wagner, PhD Thesis)

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GOAL

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I wish to explain WHY and HOW some of the theorems of domain theory and those of GMSes look astonishingly similar.

The WHY

- As noted by F. W. Lawvere both posets and GMSes are special cases of categories enriched in a closed category $\mathcal{Q}$.

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- Thus all results available for $\mathcal{Q}$-categories when specialised to $\mathcal{Q} = \mathbf{2}$ (preorders) and $\mathcal{Q} = [0, 1]$ (GMSes) will have astonishingly similar proofs.
- Varying the logic is precisely the change between $\mathbf{2}$ and $[0, 1]$.

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- Thus all results available for $\mathcal{Q}$-categories when specialised to $\mathcal{Q} = \mathbf{2}$ (preorders) and $\mathcal{Q} = [0, 1]$ (GMSes) will have astonishingly similar proofs.
- Varying the logic is precisely the change between $\mathbf{2}$ and $[0, 1]$.

  In short, astonishing similarity is a manifestation of a common categorical structure and one should study this structure to understand connection between posets and GMSes.

In Lawvere's words:

"I noticed the analogy between the triangle inequality and a categorical composition law. Later I saw that Hausdorff had mentioned the analogy between metric spaces and posets. The poset analogy is by itself perhaps not sufficient to suggest the whole system of constructions and theorems appropriate for metric spaces but the categorical connection is."

The HOW

- We challenge Lawvere's opinion by showing that the poset analogy does suggest a whole system of construction for metric spaces.

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- We challenge Lawvere's opinion by showing that the poset analogy does suggest a whole system of construction for metric spaces.
- The reason is embarassingly simple: $\mathbf{2}$ is a retract of $[0,1]$.
- However, it has non-trivial consequences: (proofs of) theorems of domain theory can be syntactically translated to (proofs of) theorems on GMSes.

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GIRARD'S BORING TRANSLATION

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The boring translation

Let A, B be formuli of intuitionistic logic. Define:

$$
\begin{array}{lcl}
A^* & = & !A \text{ for } A \text{ atomic;} \\
(A \wedge B)^* & = & A^* \otimes B^*; \\
(A \vee B)^* & = & A^* \vee B^*; \\
(A \Rightarrow B)^* & = & !(A^* \multimap B^*); \\
\mathbf{0}^* & = & \mathbf{0}; \\
(\forall x A)^* & = & ! \bigwedge x A^*; \\
(\exists A)^* & = & \bigvee x A^*.
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Then a formula F is intuitionistically provable iff F^* is provable in LL.

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Girard calls this translation BORING and of limited interests.

The boring translation

THEOREM

For $! \colon \mathcal{Q} \to \mathcal{Q}$ he set $\mathcal{H} = \mathrm{fix}(!)$ ia a complete Heyting algebra

$$(\mathcal{H}, \sqsubseteq, \sqcap, \neg_{\mathcal{H}}, \top_{\mathcal{H}}, \mathbf{0}_{\mathcal{H}})$$

with a section-retraction pair:

$$\iota \colon \mathcal{H} \rightleftarrows \mathcal{Q} \colon !$$

$$\begin{array}{lcl@{\qquad}lcl}
\iota(a \sqcap b) & = & \iota a \otimes \iota b & \iota(\top_{\mathcal{H}}) & = & \mathbf{1} \\
\iota(a \Rightarrow b) & = & !(\iota a \multimap \iota b) & \iota(\bigsqcup A) & = & \bigvee \iota A \\
\iota(a \sqcup b) & = & \iota a \vee \iota b & \iota(\bigsqcap A) & = & !(\bigwedge \iota A) \\
\iota(\neg_{\mathcal{H}} a) & = & !(\neg \iota a) & \top_{\mathcal{H}} \sqsubseteq a & \text{iff} & \mathbf{1} \leqslant \iota a \\
\iota(\mathbf{0}_{\mathcal{H}}) & = & \mathbf{0} & \mathbf{1} \leqslant !x & \text{iff} & \mathbf{1} \leqslant x.
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$$\iota\colon \mathbf{2} \rightleftarrows \mathcal{Q} \colon \texttt{ext}$$

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\iota(a \Rightarrow b) & = & \texttt{ext}(\iota a \multimap \iota b) & \iota(\bigsqcup A) & = & \bigvee \iota A \\
\iota(a \sqcup b) & = & \iota a \vee \iota b & \iota(\bigsqcap A) & = & \texttt{ext}(\bigwedge \iota A) \\
\iota(\neg_{\mathcal{H}} a) & = & \texttt{ext}(\neg \iota a) & & & \\
\iota(\mathbf{0}_{\mathcal{H}}) & = & \mathbf{0} & \mathbf{1} \leqslant \texttt{ext}(x) & \text{iff} & \mathbf{1} \leqslant x.
\end{array}$$

$$\texttt{ext}(a) := \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \bot & \text{otherwise.} \end{cases}$$

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$$\iota : \mathbf{2} \longrightarrow \mathcal{Q}$$

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\iota(a \sqcup b) & = & \iota a \vee \iota b & \iota(\bigsqcap A) & = & \bigwedge \iota A \\
\iota(\neg a) & = & \neg \iota a & & & \\
\iota(\bot) & = & \bot & \mathtt{ext} \circ \iota & = & id.
\end{array}$$

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$$\iota : \mathit{Var}(\mathbf{2}) \longrightarrow \mathit{Var}(\mathcal{Q})$$

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... and extend it to these Boolean logic rules which remain valid LL rules after the ι-translation, e.g.

$$\iota\left(\frac{a \sqcap b \sqsubseteq c}{\overline{a \sqsubseteq b \Rightarrow c}}\right) = \frac{\iota a \otimes \iota b \leqslant \iota c}{\overline{\iota a \leqslant \iota b \multimap \iota c}},$$

... and to proof trees.

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THEOREM Let $\mathcal{Q}$ be a Girard quantale.

If p is a $\mathcal{R}$-proof that $a \sqsubseteq b$ in $\mathbf{2}$, then
ιp is a $\iota(\mathcal{R})$-proof of $\iota a \leqslant \iota b$ in $\mathcal{Q}$.

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DEFINITION A proof of $x \leqslant y$ in $\mathcal{Q}$ is BORING if it is ι-translated.

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Translating domain theory to LL

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Translating order

Let X be a poset and let $X(-,-)\colon X \times X \to \mathbf{2}$ be the characteristic map of its order. Then:

(r) $\mathbf{1} \sqsubseteq X(x,x)$

(t) $\mathbf{1} \sqsubseteq (X(x,y) \sqcap X(y,z)) \Rightarrow X(x,z)$

(a) $\mathbf{1} \sqsubseteq X(x,y)$ and $\mathbf{1} \sqsubseteq X(y,x)$ imply $x = y$.

are axioms for the order.

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(t) $\mathbf{1} \leqslant (\iota X(x,y) \otimes \iota X(y,z)) \multimap \iota X(x,z)$

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- DEF. Call a pair $(X, X(-,-))$ a $\mathcal{Q}$-poset.

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is the boring translation of the order axioms.

- DEF. Call a pair $(X, X(-,-))$ a $\mathcal{Q}$-poset.
- For $\mathcal{Q} = [0,1]$ the above are quasi-metric axioms!

Translating lower subsets

- For a subset $A \subseteq X$ of a poset $(X, \sqsubseteq)$, A is lower if

$$\forall x \forall y \, [(y \in A \sqcap x \sqsubseteq y) \;\Rightarrow\; x \in A]$$

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- The ι-translation:

$$\forall x \; \forall y \;\; [\mathbf{1} \leqslant ((A(y) \otimes X(x, y)) \multimap A(x))],$$

where $A \colon X \to \mathcal{Q}$ is the ι-translation of the characteristic map of the subset A.

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- DEF. $A \colon X \to \mathcal{Q}$ is a lower in a $\mathcal{Q}$-poset X if

$$\forall x \forall y \, [X(x, y) \leqslant A(y) \multimap A(x)].$$

Translating Scott-opens

- A subset $A \subseteq X$ is Scott-open if for any $\phi \in \mathcal{I}X$:

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- Defining $H(x) := \neg A(x)$ and negating both sides:

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$$H(\mathcal{S}\phi) = \bigwedge_x (\phi(x) \multimap H(x)).$$

- In $\mathbf{2}$ this means that a subset H has the property that $\mathcal{S}\phi \in H$ iff $\phi \subseteq H$, which is exactly the definition of a Scott-closed subset H.

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Continuous $\mathcal{Q}$-posets

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Auxilary mappings

- A mapping $v\colon X \times X \to \mathcal{Q}$ is *auxiliary*, if for all $x, y, z, t \in X$:
  (i) $v(x, y) \sqsubseteq X(x, y)$.
  (ii) $X(x, y) \otimes v(y, z) \otimes X(z, t) \sqsubseteq v(z, t)$.

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- $\mathrm{Aux}(X) \ni v \mapsto \lambda x.v(-,x)\colon X \to \widehat{X}$.

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- $\mathrm{Aux}(X) \ni v \mapsto \lambda x.v(-, x)\colon X \to \widehat{X}$.
- A way-below mapping is the function $\mathbf{w}\colon X \times X \to \mathcal{Q}$

$$\mathbf{w}(x, y) := \bigwedge_{\phi \in A} (X(y, \mathcal{S}\phi) \multimap \phi x)$$

where A is the set of all ideals on X that have suprema.

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- A mapping $v\colon X \times X \to \mathcal{Q}$ is *auxiliary*, if for all $x, y, z, t \in X$:
  (i) $v(x,y) \sqsubseteq X(x,y)$.
  (ii) $X(x,y) \otimes v(y,z) \otimes X(z,t) \sqsubseteq v(z,t)$.
- $\mathrm{Aux}(X) \ni v \mapsto \lambda x.v(-,x)\colon X \to \widehat{X}$.
- A way-below mapping is the function $\mathbf{w}\colon X \times X \to \mathcal{Q}$

$$\mathbf{w}(x,y) := \bigwedge_{\phi \in A} (X(y, \mathcal{S}\phi) \multimap \phi x)$$

  where A is the set of all ideals on X that have suprema.
- The way-below mapping is auxiliary.

The way-below map

PROPOSITION. The way-below map is interpolative: for all $x, y \in X$

$$\mathbf{w}(x,y) = \bigvee_{z \in X} \left(\mathbf{w}(x,z) \otimes \mathbf{w}(z,y)\right)$$

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- THEOREM A Scott-continuous retract of a continuous $\mathcal{Q}$-domain is a continuous $\mathcal{Q}$-domain.

$\mathcal{Q}$-powerdomains

Let X be a continuous $\mathcal{Q}$-domain.

- The set $\mathcal{P}_f(X)$ of all finite subsets of X can be transformed into a $\mathcal{Q}$-preorder in a few ways:

$$\begin{array}{lclclcl}
\mathbf{H}(M,N) & := & \bigwedge_{x\in M}\bigvee_{y\in N} X(x,y); & \mathbf{h}(M,N) & := & \bigwedge_{x\in M}\bigvee_{y\in N} \mathbf{w}(x,y) \\
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- DEFINITION The Hoare (respectively: Smyth, Plotkin) $\mathcal{Q}$-powerdomain of X is the rounded ideal completion of the $\mathcal{Q}$-abstract basis $(\mathcal{P}_f(X), \mathbf{h})$ (respectively: $(\mathcal{P}_f(X), \mathbf{s})$, $(\mathcal{P}_f(X), \mathbf{p})$).

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- THEOREM The Hoare (resp.: Smyth, Plotkin) $\mathcal{Q}$-powerdomain of a continuous $\mathcal{Q}$-domain is again a continuous $\mathcal{Q}$-domain.

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THE END

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