On Domain Theory over Girard Quantales

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Keywords

- A **GMS** (generalized metric space) is a set with a distance mapping of type $X \times X \rightarrow [0, 1]$ satisfying some of the usual metric axioms.

- We can further generalize distance to type $X \times X \rightarrow Q$, where $Q$ is a Girard quantale.
Girard quantales

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- tensor: $\otimes: \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{Q}$ – associative, commutative,
Girard quantales

- A Girard quantale is a complete lattice \((Q, \leq)\) with:
  - **tensor:** \(\otimes: Q \times Q \to Q\) – associative, commutative,
  - \[ a \otimes \bigvee S = \bigvee_{s \in S} (a \otimes s), \]
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\]

- **Def.:** \(a \otimes x \leq b \iff a \leq b \to x,\)
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- **Def.:** \(a \otimes x \leq b \iff a \leq b \rightarrow x\),

- \(a = \neg\neg a\,\text{where } \neg a := a \rightarrow \bot\,\text{and } \bot\text{ is the least element,}\)
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- **Def.:** \(a \otimes x \leq b \iff a \leq b \implies x,\)

\[ a = \neg\neg a, \text{ where } \neg a := a \implies \bot, \text{ and } \bot \text{ is the least element}, \]

- **unit:** \(1 := \neg\bot,\)
A Girard quantale is a complete lattice $(Q, \leq)$ with:

- **Tensor**: $\otimes : Q \times Q \rightarrow Q$ – associative, commutative,

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- **Def.**: $a \otimes x \leq b \iff a \leq b \rightarrow x$,

- $a = \neg\neg a$, where $\neg a := a \rightarrow \bot$, and $\bot$ is the least element,

- **Unit**: $1 := \neg \bot$,

- **Par**: $a \& b := \neg(\neg a \otimes \neg b)$,
A Girard quantale is a complete lattice \((Q, \leq)\) with:

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- **Def.**: \(a \otimes x \leq b \iff a \leq b \to x,\)

- \(a = \neg\neg a,\) **where** \(\neg a := a \to \bot,\) **and** \(\bot\) **is the least element,**

- **unit**: \(1 := \neg\bot,\)

- **par**: \(a \oslash b := \neg(\neg a \otimes \neg b),\)

- **Informally**: \(\land, \lor, \otimes, \oslash, \to, \bigvee, \bigwedge, 1, \bot, \neg, !, ?.\)
Examples

Every complete Boolean algebra is a Girard quantale with $\otimes = \wedge$, e.g.:

\[
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (bot) at (0,-2) {$\bot$};
  \draw (1) -- (bot);
\end{tikzpicture}
\]
Examples

- Every complete Boolean algebra is a Girard quantale with $\otimes = \wedge$, e.g.: 

  \[ 1 \]
  \[ \perp \]

  \[ \begin{array}{c}
  1 \\
  \end{array} \]

- The two-element lattice $2 = \{1, \perp\}$ with $\otimes = \wedge$. 

On Domain Theory over Girard Quantales – p. 5/3
Examples

- Every complete Boolean algebra is a Girard quantale with $\otimes = \land$, e.g.:

\[ \begin{array}{c}
1 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
1
\end{array} \]

- The two-element lattice $2 = \{1, \bot\}$ with $\otimes = \land$.
- The unit interval $([0, 1], \geq)$ with $\otimes = +$. 
MOTIVATION
Generalized Metric Spaces

Perhaps the theory of GMSes is not as much concerned with generalizing metric spaces as with generalizing dcpos and domains:
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are devoted to solving recursive domain equations in GMSes.
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speak about generalized Alexandroff and Scott topologies.
Generalized Metric Spaces

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Perhaps the theory of GMSes is not as much concerned with generalizing metric spaces as with generalizing dcpos and domains:


proposes powerdomains for GMSes.
Generalized Metric Spaces

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- the theory is developed towards applications in denotational semantics;
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This situation is not surprising, since:

- the theory is developed towards applications in denotational semantics;
- the theorems of Scott’s domain theory are universal and prone to generalizations.
On the inverse limit construction

“The pre-order version was discovered first [...]. The metric version was mainly developed by P. America and J. Rutten.

The proofs look astonishingly similar but until now the preconditions for the pre-order and the metric versions have seemed to be fundamentally different.

In this thesis we indicate how to use one and the same proof for both cases, just varying the logic to move from one setting to the other.”

(K.R. Wagner, PhD Thesis)
GOAL
I wish to explain *WHY* and *HOW* some of the theorems of domain theory and those of GMSes **look astonishingly similar**.
As noted by F. W. Lawvere both posets and GMSes are special cases of categories enriched in a closed category $Q$. 

The WHY
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- As noted by F. W. Lawvere both posets and GMSes are special cases of categories enriched in a closed category $\mathcal{Q}$.
- Thus all results available for $\mathcal{Q}$-categories when specialised to $\mathcal{Q} = 2$ (preorders) and $\mathcal{Q} = [0, 1]$ (GMSes) will have astonishingly similar proofs.
As noted by F. W. Lawvere both posets and GMSes are special cases of categories enriched in a closed category $Q$. Thus all results available for $Q$-categories when specialised to $Q = 2$ (preorders) and $Q = [0, 1]$ (GMSes) will have astonishingly similar proofs. Varying the logic is precisely the change between 2 and [0, 1].
The WHY

As noted by F. W. Lawvere both posets and GMSes are special cases of categories enriched in a closed category $\mathcal{Q}$.

Thus all results available for $\mathcal{Q}$-categories when specialised to $\mathcal{Q} = 2$ (preorders) and $\mathcal{Q} = [0, 1]$ (GMSes) will have astonishingly similar proofs.

Varying the logic is precisely the change between $2$ and $[0, 1]$.

In short, astonishing similarity is a manifestation of a common categorical structure and one should study this structure to understand connection between posets and GMSes.
In Lawvere’s words:

“I noticed the analogy between the triangle inequality and a categorical composition law. Later I saw that Hausdorff had mentioned the analogy between metric spaces and posets. **The poset analogy is by itself perhaps not sufficient to suggest the whole system of constructions and theorems appropriate for metric spaces but the categorical connection is.**”
We challenge Lawvere’s opinion by showing that the poset analogy does suggest a whole system of construction for metric spaces.
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The reason is embarassingly simple: 2 is a retract of [0, 1].
The HOW

- We challenge Lawvere’s opinion by showing that the poset analogy does suggest a whole system of construction for metric spaces.
- The reason is embarrassingly simple: $2$ is a retract of $[0, 1]$.
- However, it has non-trivial consequences: (proofs of) theorems of domain theory can be syntactically translated to (proofs of) theorems on GMSes.
GIRARD’S BORING TRANSLATION
The boring translation

Let $A, B$ be formulii of intuitionistic logic. Define:

$$
A^* = \neg A \text{ for } A \text{ atomic;}
$$

$$(A \land B)^* = A^* \otimes B^*;$$

$$(A \lor B)^* = A^* \lor B^*;$$

$$(A \Rightarrow B)^* = \neg (A^* \rightarrow B^*);$$

$$(\forall x A)^* = \neg \bigwedge x A^*;$$

$$(\exists A)^* = \bigvee x A^*. $$

Then a formula $F$ is intuitionistically provable iff $F^*$ is provable in LL.
Let $A, B$ be formulii of intuitionistic logic. Define:

- $A^* = !A$ for $A$ atomic;
- $(A \land B)^* = A^* \otimes B^*$;
- $(A \lor B)^* = A^* \lor B^*$;
- $(A \Rightarrow B)^* = !(A^* \circ B^*)$;
- $0^* = 0$;
- $(\forall x A)^* = !\land xA^*$;
- $(\exists A)^* = \lor xA^*$.

Then a formula $F$ is intuitionistically provable iff $F^*$ is provable in LL.

Girard calls this translation BORING and of limited interests.
**The boring translation**

**THEOREM**

For $!: Q \to Q$ he set $H = \text{fix}(!)$ ia a complete Heyting algebra

$$(H, \subseteq, \sqcap, \neg, \top, 0)$$

with a section-retraction pair:

$$\iota: H \rightleftharpoons Q: !$$

\[
\begin{align*}
\iota(a \sqcap b) &= \iota a \otimes \iota b & \iota(\top_H) &= 1 \\
\iota(a \Rightarrow b) &= !(\iota a \rightarrow \iota b) & \iota(\sqcup A) &= \bigvee \iota A \\
\iota(a \sqcup b) &= \iota a \lor \iota b & \iota(\sqcap A) &= !(\sqcap \iota A) \\
\iota(\neg_H a) &= !(\neg \iota a) & \top_H \sqsubseteq a \iff 1 \leq \iota a \\
\iota(0_H) &= 0 & 1 \leq {!}x \iff 1 \leq x.
\end{align*}
\]
The boring translation

\( \iota: \mathcal{H} \rightleftharpoons \mathcal{Q}: ! \)

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\iota(a \sqcap b) &= \iota a \otimes \iota b & \iota(\top_\mathcal{H}) &= 1 \\
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\iota(a \sqcup b) &= \iota a \vee \iota b & \iota(\sqcap A) &= !(\Lambda \iota A) \\
\iota(\neg_\mathcal{H} a) &= !(\neg \iota a) & \top_\mathcal{H} \sqsubseteq a & \iff & 1 \leq \iota a \\
\iota(0_\mathcal{H}) &= 0 & 1 \leq !x & \iff & 1 \leq x.
\end{align*}
\]
The boring translation

\( \iota : 2 \leftrightarrow Q : \text{ext} \)

\[
\begin{align*}
\iota(a \sqcap b) &= \iota a \otimes \iota b \\
\iota(a \Rightarrow b) &= \text{ext}(\iota a \rightarrow \iota b) \\
\iota(a \sqcup b) &= \iota a \lor \iota b \\
\iota(\neg_{\mathcal{H}} a) &= \text{ext}(\neg \iota a) \\
\iota(0_{\mathcal{H}}) &= 0
\end{align*}
\]

\( \iota(\top_{\mathcal{H}}) = 1 \)

\( \iota(\sqcup A) = \lor \iota A \)

\( \iota(\sqcap A) = \text{ext}(\land \iota A) \)

\( 1 \leq \text{ext}(x) \iff 1 \leq x. \)

\[
\text{ext}(a) := \begin{cases} 
1 & \text{if } a = 1, \\
\bot & \text{otherwise.}
\end{cases}
\]
The boring translation

\[ \iota: \mathbf{2} \rightarrow \mathbf{Q} \]

\[
\begin{align*}
\iota(a \land b) &= \iota a \otimes \iota b \\
\iota(a \Rightarrow b) &= \iota a \rightarrow \iota b \\
\iota(a \sqcup b) &= \iota a \lor \iota b \\
\iota(\neg a) &= \neg \iota a \\
\iota(\bot) &= \bot \\
\iota(1) &= 1 \\
\iota(\sqcup A) &= \lor \iota A \\
\iota(\bigcap A) &= \land \iota A \\
\text{ext} \circ \iota &= \text{id.}
\end{align*}
\]
The boring translation

\[ \iota : \text{Var}(\mathbb{2}) \to \text{Var}(\mathcal{Q}) \]

\[
\begin{align*}
\iota(a \sqcap b) & = \iota a \otimes \iota b & \iota(1) & = 1 \\
\iota(a \Rightarrow b) & = \iota a \to \iota b & \iota(\bigsqcup A) & = \bigsqcup \iota A \\
\iota(a \sqcup b) & = \iota a \lor \iota b & \iota(\bigsqcap A) & = \bigsqcap \iota A \\
\iota(\neg a) & = \neg \iota a \\
\iota(\perp) & = \perp
\end{align*}
\]
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\[ \iota : \text{Var}(2) \rightarrow \text{Var}(Q) \]

\[
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\iota(a \Rightarrow b) & = \iota a \multimap \iota b & \iota(\bigsqcup A) & = \bigvee \iota A \\
\iota(a \sqcup b) & = \iota a \lor \iota b & \iota(\bigsqcap A) & = \bigwedge \iota A \\
\iota(\neg a) & = \neg \iota a \\
\iota(\bot) & = \bot
\end{align*}
\]

... and extend it to these Boolean logic rules which remain valid LL rules after the \( \iota \)-translation, e.g.

\[
\iota \left( \frac{a \sqcap b \sqsubseteq c}{a \sqsubseteq b \Rightarrow c} \right) = \frac{\iota a \otimes \iota b \leq \iota c}{\iota a \leq \iota b \multimap \iota c},
\]

... and to proof trees.
The boring translation

Let $\mathcal{R}$ be the collection of all Boolean logic rules that remain valid LL rules after the $\iota$-translation.
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**THEOREM** Let $\mathcal{Q}$ be a Girard quantale.

If $p$ is a $\mathcal{R}$-proof that $a \sqsubseteq b$ in $2$, then $\iota p$ is a $\iota(\mathcal{R})$-proof of $\iota a \leq \iota b$ in $\mathcal{Q}$.
The boring translation

Let $\mathcal{R}$ be the collection of all Boolean logic rules that remain valid LL rules after the $\iota$-translation.

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**THEOREM**  Let $\mathcal{Q}$ be a Girard quantale.

If $p$ is a $\mathcal{R}$-proof that $a \sqsubseteq b$ in $2$, then

$\iota p$ is a $\iota(\mathcal{R})$-proof of $\iota a \leq \iota b$ in $\mathcal{Q}$.

**DEFINITION** A proof of $x \leq y$ in $\mathcal{Q}$ is **BORING** if it is $\iota$-translated.
Translating domain theory to LL
Translating order

Let $X$ be a poset and let $X(-, -): X \times X \to 2$ be the characteristic map of its order. Then:

(r) $1 \sqsubseteq X(x, x)$

(t) $1 \sqsubseteq (X(x, y) \sqcap X(y, z)) \Rightarrow X(x, z)$

(a) $1 \sqsubseteq X(x, y)$ and $1 \sqsubseteq X(y, x)$ imply $x = y$. These are axioms for the order.
Translating order

Let $X$ be a poset and let $X(-, -): X \times X \to 2$ be the characteristic map of its order. Then:

(r) $1 \leq \iota_X(x, x)$

(t) $1 \leq (\iota_X(x, y) \otimes \iota_X(y, z)) \rightarrow \iota_X(x, z)$

(a) $1 \leq \iota_X(x, y)$ and $1 \leq \iota_X(y, x)$ imply $x = y$.

is the boring translation of the order axioms.
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Translating order

Let $X$ be a poset and let $X(-,-) : X \times X \to 2$ be the characteristic map of its order. Then:

(r) $1 \leq X(x,x)$

(t) $1 \leq (X(x,y) \otimes X(y,z)) \mapsto X(x,z)$

(a) $1 \leq X(x,y)$ and $1 \leq X(y,x)$ imply $x = y$.

is the boring translation of the order axioms.

DEF. Call a pair $(X, X(-,-))$ a $Q$-poset.
Translating order

Let $X$ be a poset and let $X(-, -): X \times X \to 2$ be the characteristic map of its order. Then:

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is the boring translation of the order axioms.

**DEF.** Call a pair $(X, X(-, -))$ a $Q$-poset.

**For** $Q = [0, 1]$ the above are quasi-metric axioms!
Translating lower subsets

For a subset $A \subseteq X$ of a poset $(X, \sqsubseteq)$, $A$ is lower if

$$
\forall x \forall y \ [ (y \in A \cap x \sqsubseteq y) \implies x \in A ]
$$
Translating lower subsets

For a subset $A \subseteq X$ of a poset $(X, \sqsubseteq)$, $A$ is lower if

$$\forall x \forall y \ [(y \in A \cap x \sqsubseteq y) \Rightarrow x \in A]$$

The $\iota$-translation:

$$\forall x \forall y \ [1 \leq ((A(y) \otimes X(x, y)) \rightarrow A(x))],$$

where $A : X \rightarrow Q$ is the $\iota$-translation of the characteristic map of the subset $A$. 
Translating lower subsets

For a subset $A \subseteq X$ of a poset $(X, \sqsubseteq)$, $A$ is lower if

$$\forall x \forall y \ [y \in A \land x \sqsubseteq y \Rightarrow x \in A]$$

The $\iota$-translation:

$$\forall x \forall y \ [1 \leq ((A(y) \otimes X(x, y)) \rightarrow A(x))]$$

where $A : X \rightarrow Q$ is the $\iota$-translation of the characteristic map of the subset $A$.

**DEF.** $A : X \rightarrow Q$ is a lower in a $Q$-poset $X$ if

$$\forall x \forall y \ [X(x, y) \leq A(y) \rightarrow A(x)].$$
A subset $A \subseteq X$ is Scott-open if for any $\phi \in \mathcal{I}X$:

$$S\phi \in A \iff (\exists x \in \phi \ (x \in A)).$$
A subset $A \subseteq X$ is Scott-open if for any $\phi \in \mathcal{I}X$:

$$S\phi \in A \text{ iff } (\exists x \in \phi (x \in A)).$$

**DEF.** $A$ is Scott-open if for any $\phi \in \mathcal{I}X$

$$A(S\phi) = \bigvee_x (\phi(x) \otimes A(x)).$$
Translating Scott-opens

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**DEF.** $A$ is **Scott-open** if for any $\phi \in \mathcal{I}X$

$$A(S\phi) = \bigvee_x (\phi(x) \otimes A(x)).$$

Defining $H(x) := \neg A(x)$ and negating both sides:

$$H(S\phi) = \bigwedge_x (\phi(x) \rightarrow H(x)).$$
Translating Scott-opens

- A subset \( A \subseteq X \) is Scott-open if for any \( \phi \in \mathcal{I}X \):

  \[
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  \]

- **DEF.** \( A \) is Scott-open if for any \( \phi \in \mathcal{I}X \)

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  \]

- Defining \( H(x) := \neg A(x) \) and negating both sides:

  \[
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  \]

- In 2 this means that a subset \( H \) has the property that \( S\phi \in H \) iff \( \phi \subseteq H \), which is exactly the definition of a Scott-closed subset \( H \).
Continuous $Q$-posets
Auxiliary mappings

A mapping \( v: X \times X \rightarrow Q \) is auxiliary, if for all \( x, y, z, t \in X \):

(i) \( v(x, y) \sqsubseteq X(x, y) \).

(ii) \( X(x, y) \otimes v(y, z) \otimes X(z, t) \sqsubseteq v(z, t) \).
Auxiliary mappings

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- $\text{Aux}(X) \ni v \mapsto \lambda x. v(-, x) : X \to \hat{X}$. 
Auxiliary mappings

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  (i) \( v(x, y) \sqsubseteq X(x, y) \).
  
  (ii) \( X(x, y) \otimes v(y, z) \otimes X(z, t) \sqsubseteq v(z, t) \).

- \( \text{Aux}(X) \ni v \mapsto \lambda x. v(-, x) : X \to \widehat{X} \).

- A way-below mapping is the function \( w : X \times X \to Q \)

  \[
  w(x, y) := \bigwedge_{\phi \in A} (X(y, \mathcal{S}\phi) \rightarrow \phi x)
  \]

  where \( A \) is the set of all ideals on \( X \) that have suprema.
Auxiliary mappings

- A mapping $v : X \times X \rightarrow Q$ is auxiliary, if for all $x, y, z, t \in X$:
  (i) $v(x, y) \subseteq X(x, y)$.
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- $\text{Aux}(X) \ni v \mapsto \lambda x. v(-, x) : X \rightarrow \widehat{X}$.

- A way-below mapping is the function $w : X \times X \rightarrow Q$
  $$w(x, y) := \bigwedge_{\phi \in A} (X(y, S\phi) \rightarrow \phi x)$$
  where $A$ is the set of all ideals on $X$ that have suprema.

- The way-below mapping is auxiliary.
**PROPOSITION.** The way-below map is interpolative: for all \( x, y \in X \)

\[
 w(x, y) = \bigvee_{z \in X} (w(x, z) \otimes w(z, y))
\]

iff Scott-continuous: for all \( x \in X \) and \( \phi \in \mathcal{I}X \) that have suprema

\[
 w(x, \mathcal{S}\phi) = \bigvee_{z \in X} (\phi z \otimes w(x, z)).
\]
Approximating maps

**DEFINITION** An auxiliary map $v: X \rightarrow \hat{X}$ is approximating if for all $x \in X$: $vx \in I X$ and $x = S(vx)$. 
**Approximating maps**

- **DEFINITION** An auxiliary map \( v: X \rightarrow \hat{X} \) is approximating if for all \( x \in X \): \( vx \in \mathcal{I}X \) and \( x = S(vx) \).

- The way-below map is below all approximating maps, however, it is not approximating in general.
Approximating maps

DEFINITION An auxiliary map $v: X \to \hat{X}$ is approximating if for all $x \in X: vx \in \mathcal{I}X$ and $x = S(vx)$.

The way-below map is below all approximating maps, however, it is not approximating in general.

DEFINITION A $Q$-poset is continuous if its way-below map is approximating.
**Approximating maps**

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- **DEFINITION** A $Q$-poset is *continuous* if its way-below map is approximating.

- Johnstone and Joyal observe that a dcpo $P$ is continuous iff the supremum has a left adjoint.
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**THEOREM** For \( v \) auxiliary, TFAE:
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**THEOREM** For \( v \) auxiliary, TFAE:

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**Approximating maps**

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- **THEOREM** For \( v \) auxiliary, TFAE:
  1. \( v \) is approximating and Scott-continuous,
  2. \( v \) is approximating and coincides with the way-below map,
Approximating maps

**DEFINITION** An auxiliary map $v: X \to \hat{X}$ is approximating if for all $x \in X$: $vx \in \mathcal{I}X$ and $x = S(vx)$.

The way-below map is below all approximating maps, however, it is not approximating in general.

**DEFINITION** A $Q$-poset is continuous if its way-below map is approximating.

Johnstone and Joyal observe that a dcpo $P$ is continuous iff the supremum has a left adjoint.

**THEOREM** For $v$ auxiliary, TFAE:

1. $v$ is approximating and Scott-continuous,
2. $v$ is approximating and coincides with the way-below map,
3. $\mathcal{I}X(vy, \phi) = X(y, S\phi)$ for all $y \in X$ and $\phi \in \mathcal{I}X$ which have suprema.
DEFINITION A $Q$-abstract basis is a $Q$-preorder $X$ equipped with an approximating relation $\nu: X \to \mathcal{I}X$ that is interpolative.
**Rounded ideals**

- **DEFINITION** A $\mathcal{Q}$-abstract basis is a $\mathcal{Q}$-preorder $X$ equipped with an approximating relation $v : X \to \mathcal{I}X$ that is interpolative.

- **DEFINITION** An ideal $\phi \in \mathcal{I}X$ is rounded if for all $x \in X$,

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\phi x = \bigvee_{z \in X} (\phi z \otimes v(x, z)).
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**THEOREM** A Scott-continuous retract of a continuous $\mathcal{Q}$-domain is a continuous $\mathcal{Q}$-domain.
**Q-powerdomains**

Let $X$ be a continuous $Q$-domain.

The set $\mathcal{P}_f(X)$ of all finite subsets of $X$ can be transformed into a $Q$-preorder in a few ways:

\[
\begin{align*}
H(M, N) &:= \bigwedge_{x \in M} \bigvee_{y \in N} X(x, y); \\
S(M, N) &:= \bigwedge_{y \in N} \bigvee_{x \in M} X(x, y); \\
P(M, N) &:= H(M, N) \otimes S(M, N); \\
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\end{align*}
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- **DEFINITION** The Hoare (respectively: Smyth, Plotkin) $Q$-powerdomain of $X$ is the rounded ideal completion of the $Q$-abstract basis $(\mathcal{P}_f(X), h)$ (respectively: $(\mathcal{P}_f(X), s)$, $(\mathcal{P}_f(X), p)$).
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**THEOREM** The Hoare (resp.: Smyth, Plotkin) $Q$-powerdomain of a continuous $Q$-domain is again a continuous $Q$-domain.
THE END