Bi-Continuous Domains and Some Old Problems in Domain Theory

Talk at Domains IX

Klaus Keimel*

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Warning: These Notes contain the contents of my Talk at Domains IX. There may be mistakes. References are incomplete. Comments are welcome.

1 Introduction

K. Martin and P. Panangaden [16] have made the interesting observation that in general relativity theory people have been dealing with partial orders and the way-below relation. From the order theoretical structure of models of spacetime which are called hyperbolic they extract the notion of a strongly hyperbolic poset. In Section 2 I will present these notion, although with a different terminology and I give a number of basic properties of these posets.

In section 3 I consider vector space orderings on $\mathbb{R}^n$. These correspond bijectively to (convex, pointed) cones C in $\mathbb{R}^n$. If these cones are closed and have inner points, then $\mathbb{R}^n$ becomes a strongly hyperbolic poset in the above sense. As there is nothing hyperbolic about these examples I prefer to adopt another terminology, although the term *strongly hyperbolic* sounds impressive.

It seems worthwile to investigate these orders on $\mathbb{R}^n$ from the point of view of domain theory. In fact, some old unsolved problems of domain theory may find their solution in this setting, as I will indicate.

Every continuous poset can be embedded in a continuous dcpo via the round ideal completion. It seems that the round ideals of the vector space orderings on $\mathbb{R}^n$ depend essentially on the geometry of the positive cone. The same is true for the dual notion of a round filter completion. It is desirable to combine these two completions. For this I propose a procedure extending a construction of Martin and Pananagaden. It would be nice to investigate whether this completion is a compactification of $\mathbb{R}^n$ and embeds $\mathbb{R}^n$ in a manifold with boundary. The same procedure can be applied to strongly hyperbolic models of spacetime, but the case of $\mathbb{R}^n$ maybe easier to manage before attacking the more general setting.

In Section 7 I change the subject completely and I add a problem on the probabilistic powerdomain over stably compact spaces.

*Fachbereich Mathematik, Technische Universität Darmstadt, 64342 Darmstadt, Germany.
email: keimel(at)mathematik.tu-darmstadt.de

We need the technical following notions for a dcpo D:

Definition . *A Scott-continuous function $f\colon D \in D$ is* finitely separated from the identity *if there is a finite set $F \subseteq D$ such that for every $x \in D$ there is a $y \in F$ such that $f(x) \leq y \leq x$. If there is a directed family (f_i) of functions f_i which are finitely separated from the identity with* $\mathrm{id} = \sup_i f_i$, *then D is called an* FS-domain.

Definition . *D is* bifinite *if it has a directed family of Scott-continuous retractions ρ_i with finite range such that* $\mathrm{id} = \sup_i \rho_i$.

One can show (see [7]: A dcpo D is a Scott-continuous retract of a bifinite domain if it has a directed family δ_i of Scott-continuous maps with finite image such that $\mathrm{id} = \sup_i \delta_i$.

2 Bi-continuous posets

Recall that in a poset P the way-below relation $a \ll b$ is defined as follows: Whenever D is a directed subset which has a least upper bound with $b \leq \bigvee^{\uparrow} D$, then $a \leq d$ for some $d \in D$. One says that P is a continuous poset if, for every $b \in P$ the set $\twoheaddownarrow b = \{a \in P \mid a \ll b\}$ is directed and $b = \bigvee^{\uparrow} \twoheaddownarrow b$.

Dually, one can define the dual way-below relation $a \ll_d b$: Whenever F is a filtered subset which has a greatest lower bound with $a \geq \bigwedge_{\downarrow} F$, then $b \geq f$ for some $f \in F$. One says that P is a dually continuous poset if, for every $a \in P$ the set $\twoheaduparrow_d a = \{b \in P \mid a \ll_d b\}$ is filtered and $a = \bigwedge_{\downarrow} \twoheaduparrow_d a$.

The Scott topology on a bi-continuous poset has as a basis for the opens the set of the form $\twoheaduparrow a$, the dual Scott topology the sets of the form $\twoheaddownarrow_d a$. The bi-Scott topology is generated by the Scott topology and its dual; it has the *open intervals*

$$]a, b[= \twoheaduparrow a \cap \twoheaddownarrow_d b = \{x \mid a \ll x \ll_d b\}$$

as a basis for the open sets. Because of the interpolation property of the way-below relation, every element also has a neighborhood basis of *closed intervals*

$$[a, b] = \uparrow a \cap \downarrow b = \{x \mid a \leq x \leq b\}$$

The closed intervals are indeed closed for the bi-Scott topology. Thus the bi-Scott topology is regular and Hausdorff. The graph of the order relation is closed; indeed, if $a \not\leq b$,indeed, if $a \not\leq b$, choose $a' \ll a$ with $a' \not\leq b$ and then choose $b' \gg b$, $b' \not\geq a'$. Then $\twoheaduparrow a'$ and $\twoheaddownarrow b'$ are disjoint neighborhoods of a and b, respectively; thus, a bi-continuous poset with the bi-Scott topology is an ordered topological space in the sense of Nachbin [19]. Moreover, If A is a Scott-closed set and $a \notin A$, then there is a dually Scott-open set U containing A and a Scott-open set V containing a which are disjoin.

Question 2.1. *Do we have a completely regular ordered space?*

In a bi-continuous poset the way-below relation need not agree with the dual way-below relation. An easy example for this phenomenon is the powerset (X) of a infinite set X oredered by inclusion, where $A \ll B$ is equivalent to A being finite and $A \ll_d B$ is equivalent to B being finite.

Definition . *A poset is called* jointly bi-continuous *if it is bi-continuous and if the way-below relation coincides with the dual way-below relation.*[1]

A bi-continuous poset is locally compact (for the bi-Scott topology) iff each of its points has a closed interval as a neighborhood which is compact or, equivalently, if for every x there are elements $a \ll x \ll_d b$ such that the closed interval $[a, b]$ is compact.

On every continuous poset, the Lawson topology is Hausdorff. On a bi-continuous poset it is coarser than the bi-Scott topology. Thus, on the bi-Scott compact subsets of a bi-continuous poset, both topologies agree. Thus, if a bi-continuous poset is locally compact for the bi-Scott topology, the latter agrees with the Lawson topology.

Definition . *A bi-continuous poset is called interval-compact if all of its closed intervals are compact.*[2]

In an interval-compact bi-continuous poset every upper bound directed set has a supremum and every lower bounded filtered set has an infimum.

Completely distributive complete lattices are bi-continuous and compact in the bi-Scott topology, and in particular interval-compact.

Problem 2.2. *Characterise the completely distributive lattices that are jointly bi-continuous.*

It seems of interest to weaken the requirement of compact intervals to the requirement that upper bounded directed sets have a least upper bound and lower bounded filtered sets have a greatest lower bound. Under this weaker hypothesis the construction of the interval domain of Martin and Panangaden works perfectly well. It seems that there bijection between interval domains and globally hyperbolic posets extends to the slightly more general situation by omitting the last condition in their definition of an interval domain.

3 Cones and orders in $\mathbb{R}^n$

A *cone* in $\mathbb{R}^n$ is meant to be subset C with the following properties:

$$(1) \quad C \cap -C = \{0\}, \quad (2) \quad C + C \subseteq C \quad (3) \quad \mathbb{R}_+ \cdot C \subseteq C$$

If we replace property (1) by the weaker property (1') $0 \in C$, we talk about a *wedge*. For a wedge W, the set $E = W \cap -W$ is a linear subspace called the *edge* of the wedge.

The orders that we want to consider on $\mathbb{R}^n$ first are *vector space orderings* $\leq$ satisfying

$$x \leq y, r \in \mathbb{R}_+ \implies x + z \leq y + z, \ rx \leq ry$$

[1]Martin and Panangaden [16] use the term *bi-continuous* for what we call *jointly bi-continuous*. They also use a different but equivalent definition: (1) P is a continuous poset, (2) whenever $a \ll b$ and F is a filtered set with $\bigwedge_{\downarrow} F \leq a$, then $f \leq b$ for some $f \in F$, and (3) $\twoheaddownarrow$a is filtered and $\bigwedge_{\downarrow} \twoheaddownarrow = a$ for every a . One also should notice that the notion of a *linked bi-continuous poset* as introduced in [4] is different from joint bi-continuity.

[2]Interval-compact bi-continuous posets have been called strongly hyperbolic posets by Martin and Panangaden [16]. The reason is that they made the interesting observation that they occur in general relativity theory in models of spacetime called strongly hyperbolic there. Although this terminology sounds great, we do not want to adopt it. In view of the examples that we will discuss in the next sections the term *strongly hyperbolic* is inappropriate. There is nothing justifying the term hyperbolic in the definition and or in the examples.

There is a bijective correspondence between cones and vector space orderings. The correspondence is given in the following way: For every cone C, we define $\leq_C$ by $x \leq_C y$ iff $y - x \in C$ iff $y \in x + C$ and, for every vector space ordering $\leq$, we consider its positive cone $C_{\leq} = \{x \in \mathbb{R}^n \mid x \geq 0\}$.

We fix a positive cone C in $\mathbb{R}^n$ and the associated order $\leq$. We denote by C° the algebraic interior of C which can be characterised as the topological interior of C in the linear subspace $V = C - C$ of $\mathbb{R}^n$.

Properties 3.1. *(a) $x \mapsto -x$ is an order anti-isomorphism.*

(b) $\mathbb{R}^n$ is directed with respect to $\leq$ iff the positive cone C generates $\mathbb{R}^n$ (i.e., $\mathbb{R}^n = C - C$) iff the interior of C in $\mathbb{R}^n$ is nonempty.

(c) C *is closed* $\iff$ *the graph of C is closed in $R^n \times \mathbb{R}^n$*
$\iff$ *every upper bounded directed set has a supremum*
$\iff$ *every lower bounded filtered set has an infimum.*

(d) $C^\circ + C \subseteq C^\circ$ and $C^\circ \cup \{0\}$ is a cone.

Properties 3.2. *Suppose now that C is a closed cone. We endow $\mathbb{R}^n$ with its usual topology and the associated order $\leq$.*

(a) $\mathbb{R}^n$ is jointly bi-continuous and $x \ll y$ iff $y \in x + C^\circ$ iff $x \in y - C^\circ$ iff $y \gg x$, i.e., the way-below relation is the strict order associated with the cone $C^\circ \cup \{0\}$ and it coincides with the dual way-below relation.

(b) $x \ll y$ and $0 < r \in \mathbb{R} \implies x + z \ll y + z$ and $rx \ll ry$, i.e., the reflexive hull of $\ll$ is a vector space ordering.

(c)$\mathbb{R}^n$ is interval-compact in the bi-Scott topology.

(d) Suppose that the cone C generates $\mathbb{R}^n$. Then the Scott-open sets are the open upper sets, the dually Scott-open sets are the open lower sets. The bi-Scott topology coincides with the usual topology.

(e) The Lawson topology as well as the dual Lawson topology coincide with the usual topology on $\mathbb{R}^n$.

(f) $\mathbb{R}^n$ is not linked bi-continuous, i.e., the interval topology (which has the closed intervals as a basis for the closed sets) does not agree with the usual topology; in fact, the interval topology is not T_0.

Proposition 3.3. *Addition on $\mathbb{R}^n$ is Scott-continuous and dually Scott-continuous.*

On a jointly bi-continuous poset we consider the bi-Scott topology which is the topology generated by the Scott topology and the dual Scott topology. It has a basis of *open intervals*

$$]a, b[= \twoheaduparrow a \cap \twoheaddownarrow b = \{x \mid a \ll x \ll b\}$$

Because of the interpolation property, every point also has a neighborhood basis of *closed intervals*

$$[a, b] = \uparrow a \cap \downarrow b = \{x \mid a \leq x \leq b\}$$

Notice that closed intervals are indeed closed in the bi-Scott topology.

The bi-Scott topology is Hausdorff. The graph of the order is closed; indeed, if $a \not\leq b$, choose $a' \ll a$ with $a' \not\leq b$ and then choose $b' \gg b$, $b' \not\geq a'$. Then $\twoheaduparrow a'$ and $\twoheaddownarrow b'$ are disjoint neighborhoods of a and b, respectively.

On $\mathbb{R}^n$ we can define lots of orders of this type via cones or more generally by slightly generalizing cones:

Let C be a subset of $\mathbb{R}^n$ with the following properties:

$$(1) \quad C \cap -C = \{0\}, \quad (2) \quad C + C \subseteq C$$

These sets C are in a bijective correspondence with the translation invariant orders on $\mathbb{R}^n$ given as above: For given C define $x \leq_C y$ iff $y - x \in C$. Again this order is closed iff C is closed.

We consider special sets C of this type: Let $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a continuous function with the following properties:

$$(1) \quad 0 = f(0) < f(x) + f(-x), \quad (2) \quad f(x + y) \leq f(x) + f(y) \text{ (sub-additivity)}$$

Let $C \subseteq \mathbb{R}^n$ be the epigraph of the function F, that is,

$$C = \{(x, r) \mid f(x) \leq r\}$$

Property (1) is equivalent to $C \cap -C = \{0\}$ and property (2) is equivalent to $C + C \subseteq C$; moreover C is closed. Note that C is a cone iff f is positively homogeneous, in addition. The interior of C is the set

$$C^\circ = \{(x, r) \mid f(x) > r\}$$

Again, $C^\circ + C \subseteq C^\circ$. With the order relation $\leq$ associated with C, $\mathbb{R}^n$ becomes a jointly bi-continuous poset with $x \ll y$ iff $y \in x + C^\circ$ iff $x \in y - C^\circ$ iff $y \gg x$.

In general, the bi-Scott topology may be coarser than the usual topology on $\mathbb{R}^n$. But as both are Hausdorff, they agree on the bounded subsets of $\mathbb{R}^n$.

Closed intervals need not be compact in the bi-Scott topology (Example: Take the epigraph of cissoid of Diocles given by the equation $(y-1)x^2 = y^3$). When all closed intervals are compact, then the bi-Scott topology coincides with the usual one.

It might be that 'small' closed intervals are compact?

Similar situations as those sketched above have been considered in classical mathematics. One may replace $\mathbb{R}^n$ by a smooth manifold M. We have already cited the paper of Martin and Panangaden [16],where they consider space-time manifolds with the causal order. Quite more generally, Lawson has introduced orders on manifolds induced by cone fields. If we assign to every point x of a smooth manifold M a cone $C(x)$ in the tangent space of M at x, we have a *cone field*. We write $x \prec y$ if there is a continuous piecewise smooth curve from x to y whose derivative (tangent) belongs to the tangent cone $C(z)$ in each of its smooth points. The relation $\prec$ is a preorder. We consider the closure $\leq$ of the relation $\prec$ which is again a preorder. If it is a partial order we call it the conal order and we say that M is globally ordered. In the case of homogeneous spaces, where the tangent cone is shifted around by the group action, the compactness of the intervals in the conal order is an important issue as this property is needed to define the Volterra algebra. Neeb [20] has characterised those globally orderable homogeneous spaces for which the order intervals are compact.

It seems worthwhile to look at the quite vast literature on causal orders on manifolds in view of their domain theoretical properties.

4 Cone orders on subsets of $\mathbb{R}^n$

We return to the situation of an order $\leq$ on $\mathbb{R}^n$ given by a closed generating cone. We may restrict this order to subsets A of $\mathbb{R}^n$. We only will consider convex subsets A. First a remark:

Remark 4.1. *There is a linear functional φ on $\mathbb{R}^n$ such that $\varphi(x) > 0$ for every nonzero element x of C. We denote by H the hyperplane $\varphi^{-1}(0)$ which meets C only in $\{0\}$. The set*

$$K = \{c \in C \mid \varphi(x) = 1\}$$

is a compact convex subset of C with the property that for every $x \neq 0$ in C there is a unique positive real number r such that $rx \in K$. Every such subset K is called a base *of C. The convex hull of K and $\{0\}$, i.e., the set*

$$S = \{c \in C \mid \varphi(x) \leq 1\}$$

is called a kegelspitze *of C.*

As a first case, we restrict our order to the cone $A = C$ itself. C remains bi-continuous. The way-above relation is the one induced from the way-above relation on $\mathbb{R}^n$. But the way-below relation is changed on the boundary of C. Thus $C, \leq_C$ is not jointly bi-continuous. We need some notation: For $a \in C$, let

$$C_a = \{x \in C \mid x \leq_C na \text{ for some natural number } n\} = C \cap \bigcup_n (na - C)$$

As $\bigcup_n (na - C)$ is a linear subspace of $\mathbb{R}^n$, C_a is a closed subcone and a lower set in C. Moreover, C_a is a *face* of C, that is, whenever a convex combination of elements $x, y \in C$ belongs to C_a, then x and y belong to C_a. We say that C_a is the face of C generated by a. Every face of C of this form.

Proposition 4.2. $a \ll b$ *iff* $b \in a + (C_b)^\circ$.

Problem 4.3. *The kegelspitze S is a continuous dcpo. Is it an FS-domain?*

If the basis K is a disc, then S is not an FS-domain.

Now let us consider $-C$. We may add $\perp$ as a smallest element to C. This destroys the property of being dually continuous. (Going down to $\perp$ along an extreme ray of $-C$ shows that $\perp \not\ll 0$.) We claim:

Proposition 4.4. $-C_\perp$ *ordered by* $\leq_C$ *is an FS-domain.*

Proof. Let a be an element in the interior of C with $\varphi(a) = 1$. For every n, define $f_n\colon -C_\perp \to -C_\perp$ in the following way: $f_n(x) = x - \frac{1}{n}a$, if $\varphi(x) > n$, and $f_n(x) = \perp$, else. Then f_n is an increasing sequence of Scott-continuous functions such that $\sup_n f_n = \mathrm{id}$. Moreover, each f_n is finitely separating. One has to show that there is a finite set F such that for every $x \in C$, there is a $y \in F$ such that $f(x) \leq y \leq x$. This can be done. $\square$

Problem 4.5. *(Jung, Lawson) Is $-C_\perp$ a retract of a bifinite domain?*

Now let D be a closed cone properly containing the cone C. Then D is not a continuous poset with respect to $\leq_C$. Indeed, D has an extreme ray R which does not belong to C. The points a on the extreme ray are minimal in D with respect to $\leq_C$ but not compact.

At the other hand, if D is a closed cone contained in C, then D is a continuous poset with respect to $\leq_C$. The way-below relation is the one induced by the way-below relation on C.

5 The probabilistic powerdomain

Let us concentrate on the probabilistic powerdomain over a finite poset P. We denote by $\mathcal{O}P$ the the collection of all upper sets $U \subseteq P$. Ordered by inclusion, $\mathcal{O}P$ is a finite distributive lattice, and every finite distributive lattice L occurs in this way, namely L is (isomorphic to) the collection of upper sets of the set P of $\vee$-irreducible elements of L.

In the vector space $\mathbb{R}^P$ we consider the polyhedral cone C defined by the inequalities

$$\sum_{p\in U} x_p \geq 0,\ U \in \mathcal{O}P$$

Clearly C contains the cone $\mathbb{R}^P_+$,

Example 5.1. *Consider a poset with four elements 1, 2, 3, 4, where 1 and 2 are incomparable, 3 and 4 are incomparable, and where 1 and 2 are both dominated by 3 and 4. The cone C in $\mathbb{R}^4$ is given by the five linear inequalities*

$$\begin{aligned} x_3 &\geq 0 \\ x_4 &\geq 0 \\ x_1 + x_3 + x_4 &\geq 0 \\ x_2 + x_3 + x_4 &\geq 0 \\ x_1 + x_2 + x_3 + x_4 &\geq 0 \end{aligned}$$

We may view $\mathbb{R}^P$ as the set of all measures on the finite set P. The order $\leq_C$ has been called the stochastic order by Edwards [2]. We may restrict the stochastic order to $\mathbb{R}^P_+$, the set of all positive measures on P, to $\mathcal{V}_{\leq 1}P$ and $\mathcal{V}_1P$, the subprobability and the probability measures, that is, the set of all $x \in \mathbb{R}^P_+$ such that $\sum_{p\in P} x_p \leq 1$ and $\sum_{p\in P} x_p = 1$, respectively.

Problem 5.2. *Is the subprobabilistic and the probabilistic powerdomain on a finite poset with the stochastic order an FS-domain? or even a retract of a bifinite domain?*

An answer is only known for very special cases only: If the poset P is a tree, then $\mathcal{V}_{\leq 1}P$ is a retract of a bifinite domain. If P is a finite root system, then $\mathcal{V}_{\leq 1}P$ is an FS-domain (see Jung and Tix [9]).

6 Compactnes of the space of causal curves

For a subset K of a poset the set $\mathsf{o}-\mathsf{conv}(K) =_{def} \downarrow K \cap \uparrow K = \bigcup_{a,b\in K}[a,b]$ is the smallest order-convex subset containing K; we call it the *order-convex hull* of K.

In this section we consider a bicontinuous interval-compact poset X and we denote be $\mathcal{C}X$ the set of all nonempty order-convex bi-Scott compact subsets of X. There are two orders of interest on $\mathcal{C}X$, the inclusion order $\subseteq$ and the Egli-Milner order $P \sqsubseteq Q$ if for every $x \in P$ there is a $y \in Q$ such that $x \leq y$ and vice-versa. As X is supposed to be interval-compact, the convex hull of any finite subset is compact.

Proposition 6.1. *Let K be a nonempty bi-Scott compact subset. Then there is a family (F_i) of finite subsets of X with the following properties: $\mathsf{o}-\mathsf{conv}(F_i)$ is a neighbourhood basis of $\mathsf{o}-\mathsf{conv}(K)$ and $\bigcap_i \mathsf{o}-\mathsf{conv}(F_i) = \mathsf{o}-\mathsf{conv}(K)$. In particular, $\mathsf{o}-\mathsf{conv}(K)$ is compact.*

Proof. (Sketch) Let U be an open neighbourhood of K. For every $z \in K$, choose an open interval neighborhood $]a'_z, b'_z[$ of z contained in U. Using interpolation choose a_z, b_z such that $a'_z \ll a_z \ll z \ll_d b_z \ll_d b'_z$. By the compactness of K, there are finitely many z_i such that the open intervals $]a_{z_i}, b_{z_i}[$ cover K. Let F be the set of these finitely many a_{z_i} and b_{z_i}. Then $\bigcup_{c,d\in F}]c,d[$ is order-convex, open and contains K, whence also the order-convex hull of K. Further the order convex hull $\mathsf{o}-\mathsf{conv}(F) = \bigcup_{c,d\in F}[c,d]$ is compact and contains the order-convex hull of K. If the open set U is order-convex, then the order-convex hull of F is also contained in U, as $F \subseteq U$.

If a is an element not in the convex hull of K, then $a \notin \downarrow K$ or $a \notin \uparrow K$. suppose for example that $a \notin \downarrow K$. Then K is contained in the open order-convex set $U = X \setminus \uparrow a$. By the first paragraph, there is a finite set F in U such that $\mathsf{o}-\mathsf{conv}(K) \subseteq \mathsf{o}-\mathsf{conv}(F) \subseteq U$, whence $a \notin \mathsf{o}-\mathsf{conv}(F)$. Thus, the order-convex hull of K is the intersection of finitely generated order convex sets which implies that $\mathsf{o}-\mathsf{conv}(K)$ is compact.

If we begin with K being already order-convex, then the above shows that K is the intersection of finitely generated order-convex set which contain K in their interior. It follows that these finitely generated order-convex sets form a neihgbourhood basis of K. □

Corollary 6.2. *Under the containment order, $\mathcal{K}X$ is a dually continuous domain. We have $A \ll_d B$ iff there is a finite set $F \subseteq B$ such that $A \subseteq \int \mathsf{o}-\mathsf{conv}(F)$.*

7 Completions

Every continuous poset P can be embedded in a continuous dcpo by taking the round ideal completion $\mathcal{J}P$. Recall that a round ideal is a directed lower set J with the property that for every $x \in J$ there is a $y \in J$ with $x \ll y$. The round ideals are ordered by inclusion. The embedding is given by $x \mapsto \mathop{\downarrow\!\!\!\downarrow} x$. In a bounded directed complete continuous poset, every upper bounded round ideal is of the form $\mathop{\downarrow\!\!\!\downarrow} x$ and the round ideal completion just adds unbounded round ideals on top.

First let us concentrate on $\mathbb{R}^n$ with n order given by some closed generating cone C. The sum of two round ideals is a round ideal and also the scalar multiple for $r > 0$.

Lemma 7.1. *The round ideal completion of $\mathbb{R}^n$ is almost a continuous d-cone ($r \to rx$ is not monotone for $x \not\geq 0$).*

But is the round ideal completion also dually continuous?

Problem 7.2. *Describe the round ideal and the round filter completion concretely.*

Example 7.3. *Consider $\mathbb{R}^2$ with the cone $C = \mathbb{R}^2_+$. The round ideals are I_{1r} given by the inequality $x_1 < r$ and I_{2r} given by the inequality $x_2 < r$, plus the whole of $\mathbb{R}^2$. The round ideal completion can be identified with $(\mathbb{R} \cup \{+\infty\})^2$ which is again dually continuous. The round filter completion of it becomes $(\mathbb{R} \cup \{-\infty\})^2$. One may consider $(\mathbb{R} \cup \{+\infty - \infty\})^2$ to be jointly a round ideal and round filter completion.*

Conjecture: The round ideals are the sets $I_{f,r}$ given as follows: Take any tangent hyperplane to the cone C, given by a linear form f with $f(C) = \mathbb{R}_+$. Then take the sets $I_{f,r} = \{x \in C \mid f(x) < r\}$ for $r \in \mathbb{R}$.

It would be nice to combine the round ideal and the round filter completion with the aim to obtain a jointly bi-continuous poset which is compact in the bi-Scott topology, as far as this is possible. One attempt to achieve this goal could consist in an appropriate extension of the interval domain construction as performed by Martin and Panangaden [16]:

They consider the set $\mathcal{I}$ of all closed intervals $[x, y]$ under reverse containment. This is the same as the set $\{(x, y) \mid x \leq y\} \in \mathbb{R}^n \times \mathbb{R}^n$ under the order $(x, y) \sqsubseteq (x', y')$ iff $x' \leq x$ and $y \leq y'$. This set is in fact a wedge. The edge of the wedge consists of the diagonal $\Delta = \{(x, x) \mid x \in L\}$ which are the minimal elements of $\mathcal{I}$. $\mathcal{I}$ is a continuous domain. The original space is the set of maximal elements with the topology induced by the Scott topology on $\mathcal{I}$.

I want to propose the following modification. It works for jointly bi-continuous posets. One may even replace the poset by any dense subset with the inherited way-below relation: Let us call *round interval* every non-empty intersection of a round filter with a round ideal. The bounded round intervals are the nonempty open intervals $]a, b[$. The collection $\mathcal{I}$ of all round intervals is ordered by inclusion. There is a second order, the Egli-Milner order $I \sqsubseteq J \iff I \subseteq \downarrow J$ and $\uparrow I \supseteq J$. We want to consider $\mathcal{I}$ as an abstract basis with the relation

$$I \ll J \iff \exists a, b \in J. I \subseteq]a, b[$$

Consider the round filter completion $\mathcal{F}(\mathcal{I})$ which is a dcpo. The set of all maximal elements of this completion is the desired bi-completion. Is the order inherited from the Egli-Milner order on the round ideals $I \sqsubseteq J \iff I \subseteq \downarrow J$ and $\uparrow I \supseteq J$.The interval domain $\mathcal{I}$ is embedded in $\mathcal{F}(\mathcal{I})$ by assigning to each closed nonempty interval $[a, b]$ the collection of all open intervals $[a', b']$ with $a' \ll a$ and $b \ll b'$, and the continuous poset itself is embedded via its embedding

Of course, we may also consider the restriction of the order $\leq_C$ to the cone C itself or to a closed subcone D of C an consider the round ideal completion which is a continuous d-cone. Again it would be nice to describe the round ideals explicitly.

Problem When is the round ideal completion of C an FS-domain?

8 Topological measure theory and d-cones

The theory of continuous valuations introduced by Lawson, Jones and Plotkin can be viewed as an attempt to develop an analogue of classical topological measure theory which concerns only Hausdorff spaces in a T_O setting. The two developments are parallel. There are parallel results with different proofs. It would be desirable to close the gap between the two theories.

The relation between valuations and Borel measures has been clarified for a large class of spaces. Let me indicate an open question.

Classically, one may associate to every compact Hausdorff space X the set $\mathcal{P}X$ of all probability measures which is a compact convex set in the space of all Borel measures with the vague topology. Associating to every compact Hausdorff space X the compact convex set $\mathcal{P}X$ yields a monad over the category sf Comp of compact Hausdorff spaces and continuouos maps. The algebras of this monad have been characterised by Swirszcz to be the compact convex sets embeddable in locally convex vector spaces. For the proof classical functional analytic tools are used quite heavily.

In domain theory, C. Jones has looked at the subprobabilistic powerdomain monad over continuous dcpos and Scott-continuous maps. She has characterised the algebras of this monad to be the continuous domains which are abstract convex sets.

We ask the question whether there is a theorem unifying these two results. We formulate the following

Problem 8.1. *Characterise the algebras over the subprobabilistic powerdomain monad over the category* Stabcomp *of stably compact spaces and continuous maps.*

As the category Comp is a full subcategory of StabComp such a characterisation would contain the classical result. It would not quite contain Jones result, as not every continuous domain is stably compact. But as most continuous domains are stably compact, such a characterisation would be quite satisfactory also for continuous dcpos. On the way towards such a charcterisation I have made some slight progress which I want to describe now.

9 Appendix: Vector space orderings on $\mathbb{R}^n$ and the formal ball model

Let me point out another interesting class of jointly bicontinuous posets.

In his talk at Domains IX, J. D. Lawson talked about the *extended formal ball model* due to Tsuiki and Hattori [26]: For a metric space X with metric d, the extended formal ball model is the set $B(X) = X \times \mathbb{R}$ with the order $(x, r) \leq (y, s)$ iff $d(x, y) \leq r - s$. For the usual formal ball model, the above domain is restricted to $B_+(X) =_{def} X \times \mathbb{R}_+$.

We reproduce some results of Tsuiki and Hattori [26]: In the extended formal ball model $B(X)$ the way-below relation is given by $x \ll y$ iff $d(x, y) < r - s$. The map $(x, r) \mapsto (x, -r)$ is an order anti-isomorphism and $B(X)$ is a jointly bi-continuous poset. The bi-Scott topology on $B(X) = X \times \mathbb{R}$ agrees with the product of the metric topology on X and the usual topology on $\mathbb{R}$. If the metric space X is totally bounded, then the upper topology equals the Scott topology and the lower topology the dual Scott topology on $B(X)$, that is, the poset is linked bicontinuous in the sense of [4].

Let us rise the question, under which condition intervals are compact for the bi-Scott topology.

As the name says, the formal balls (x, r) represent formally the concrete balls around the point x with radius r. From this point of view, it is difficult to develop an intuition for balls with negative radii. In the follwing we give a concrete representation of the formal ball model for finite dimensional normed spaces: In this case, $B(X)$ can be viewed as an $\mathbb{R}^n$ with an order given by a certain closed generating cone as in Section 3.

Let $X = \mathbb{R}^{n-1}$ with a metric is given by a norm. The usual formal ball domain $\mathbb{R}^{n-1} \times \mathbb{R}_+$ is isomorphic to the set of all concrete closed balls in $\mathbb{R}^{n-1}$. For a concrete representation of the extended formal ball domain $\mathbb{R}^{n-1} \times \mathbb{R}$, let K be the closed unit ball for the given norm. Consider in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ the cone $C = \{(rx, r) \mid r \in \mathbb{R}_+\}$ which has $K \times \{1\}$ as a basis. If we endow $\mathbb{R}^n$ with the order $x \leq y$ iff $y - x \in C$, then $\mathbb{R}^n$ becomes a poset isomorphic to the extended formal ball model $B(\mathbb{R}^{n-1})$. The isomorphism is given by the map $(x, r) \mapsto (rx, -r)$.

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