#### A Model of Cooperative Threads

#### Martín Abadi Gordon Plotkin

Microsoft Research, Silicon Valley LFCS, University of Edinburgh

Domains IX, University of Sussex 2008

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# Outline



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## **Cooperative Threads and AME**

- Cooperative Threads run without interruption until they yield control.
- Interest in such threads has increased recently with the introduction of Automatic Mutual Exclusion (AME) and the problem of programming multicore systems.

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## What we do

 We describe a simple language for cooperative threads and give it a mathematically elementary fully abstract (may) semantics of sets of traces, being *transition sequences* of, roughly, the form:

$$u = (\sigma_1, \sigma'_1) \dots (\sigma_m, \sigma'_m)$$

à la Abrahamson, the authors, Brookes etc, but adapted to incorporate thread spawning.

 Following the algebraic theory of effects, we characterise the semantics using a suitable inequational theory, thereby relating it to standard domain-theoretic notions of resumptions.

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$$b \in BExp = \dots$$

$$e \in NExp = \dots$$

$$C, D \in Com = skip$$

$$| x := e \quad (x \in Vars)$$

$$| C; D$$

$$| if b then C else D$$

$$| while b do C$$

$$| async C$$

$$| yield$$

$$| block$$

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async x := 0; x := 1; yield; if x = 0 then x := 2 else block

This spawns the asynchronous execution of x := 0, executes x := 1, yields, then resumes but blocks unless the predicate x = 0 holds, then executes x := 2

With respect to safety properties, the conditional blocking amounts to awaiting that x = 0 holds. So The last line may be paraphrased as

await 
$$x = 0; x := 2$$

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### **Operational Semantics**

$$\langle \sigma, T, \mathcal{E}[\mathbf{x} := \mathbf{e}] \rangle \langle \sigma, T, \mathcal{E}[\operatorname{skip}; \mathbf{C}] \rangle \langle \sigma, T, \mathcal{E}[\operatorname{if} b \text{ then } \mathbf{C} \text{ else } \mathbf{D}] \rangle \langle \sigma, T, \mathcal{E}[\operatorname{while} b \text{ do } \mathbf{C}] \rangle \langle \sigma, T, \mathcal{E}[\operatorname{async } \mathbf{C}] \rangle \langle \sigma, T, \mathcal{E}[\operatorname{vield}] \rangle$$

 $\langle \sigma, T.C.T', \text{skip} \rangle$ 

 $\begin{array}{ll} & \longrightarrow_a & \langle \sigma[x \mapsto \sigma(e)], T, \mathcal{E}[\mathrm{skip}] \rangle \\ & \longrightarrow_a & \langle \sigma, T, \mathcal{E}[C] \rangle \\ & \longrightarrow_a & \langle \sigma, T, \mathcal{E}[C] \rangle \\ & \quad (\mathrm{if} \ \sigma(b) = \mathrm{true}) \\ & \longrightarrow_a & \langle \sigma, T, \mathcal{E}[C; \mathrm{while} \ b \ \mathrm{do} \ C] \rangle \\ & \quad (\mathrm{if} \ \sigma(b) = \mathrm{true}) \\ & \longrightarrow_a & \langle \sigma, T. \mathcal{C}, \mathcal{E}[\mathrm{skip}] \rangle \\ & \longrightarrow_a & \langle \sigma, T. \mathcal{E}[\mathrm{skip}], \mathrm{skip} \rangle \\ & \longrightarrow_c & \langle \sigma, T. T', C \rangle \end{array}$ 

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## State Space and Evaluation Contexts

#### State Space

- $\Gamma \in$  State = Store × ThreadPool × Com
- $\sigma \in \text{Store} = \text{Vars} \rightarrow \text{Nat}$
- $T \in \text{ThreadPool} = \text{Com}^*$

where Vars is finite.

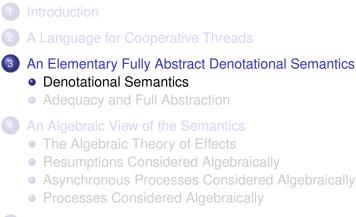
**Evaluation Contexts** 

$$\mathcal{E} = [] | \mathcal{E}; \mathcal{C}$$

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Denotational Semantics Adequacy and Full Abstraction

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## **Transition Sequences**

• Abrahamson used transition sequences of the form:

$$u = (\sigma_1, \sigma'_1) \dots (\sigma_m, \sigma'_m)$$

• Perhaps we need hierarchical triples for thread spawning:

$$\mathbf{v} = (\sigma_1, u_1, \sigma'_1) \dots (\sigma_m, u_m, \sigma'_m)$$

• Miraculously, we only need 1 embedding to 1 level, roughly:

$$\mathbf{v} = (\sigma_1, \sigma'_1) \dots (\sigma_m, u, \sigma'_m)$$

• Precisely, so that prefix is the right partial order, and also to allow for totality, *transition sequences* are:

$$\mathbf{v} = (\sigma_1, \sigma'_1) \dots (\sigma_m, \sigma'_m) [(\sigma, \sigma' \text{ return}) u]$$

where  $u = (\overline{\sigma}_1, \overline{\sigma}'_1) \dots (\overline{\sigma}_n, \overline{\sigma}'_n)$ [done] is a pure transition sequence (and  $m, n \ge 0$ ).

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# Form of Denotational Semantics

Proc, our domain of *processes*, is  $I_{\neq \emptyset, \omega}$ (TSeq) the  $\omega$ -cpo of all non-empty, countably-based ideals of transition sequences, i.e., all nonempty prefix-closed sets of transition sequences. We have:

 $\llbracket C \rrbracket \in \operatorname{Proc}$ 

Pool, our domain of *thread pools*, is  $\mathcal{I}_{\neq \emptyset, \omega}(PSeq)$  the  $\omega$ -cpo of all non-empty, countably-based ideals of pure transition sequences, i.e., the  $\omega$ -cpo of all non-empty prefix-closed sets of pure transition sequences. We have:

 $\llbracket T \rrbracket \in \text{Pool}$ 

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## **Denotational Semantics of Commands**

[skip] = \* $\llbracket C; D \rrbracket = \llbracket C \rrbracket \circ \llbracket D \rrbracket$  $[x := e] = \{(\sigma, \sigma [x \mapsto n] \text{ return}) \text{ done } | \sigma \in \text{Store}, \sigma(e) = n\} \downarrow$ [if b then C else D] =  $\{(\sigma, \tau) v \in [C] \mid \sigma(b) = \text{true}\} \downarrow \cup$  $\{(\sigma, \tau) v \in \llbracket D \rrbracket \mid \sigma(b) = \text{false}\} \downarrow$ [while b do C] =  $\bigcup_i$  [(while b do C)<sub>i</sub>] [async C] = async([C]<sup>c</sup>)[vield] = d(\*) $[block] = \{\varepsilon\}$ 

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## Sequential Composition of Processes

We give rules for composition, as it is easier to understand that way:

$$\frac{v(\sigma, \sigma' \operatorname{return})u \in P \quad (\sigma', \tau)w \in Q}{v(\sigma, \tau)(u \bowtie w) \subseteq P \circ Q}$$
$$\frac{v \in P}{v \in P \circ Q} \quad \text{if } v \text{ does not contain return}$$

It is associative with two-sided unit:

\* = {(
$$\sigma, \sigma$$
 return) done |  $\sigma \in$  Store}  $\downarrow$ 

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#### Merging transition sequences

The set of merges of a pure transition sequence u and a (pure) transition sequence w is given by:

$$u[\text{done}]^1 \bowtie w[\text{done}]^2 = (u \bowtie w)[\text{done}]^{1 \land 2}$$

where the merge on the right is the standard merge of sequences and the done on the right appears only if it appears both times on the left.

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### **Delay and Yielding**

We define a continuous *delay* function  $d : Proc \rightarrow Proc$  by:

$$d(P) = \{(\sigma, \sigma) v \mid \sigma \in \text{Store}, v \in P\} \downarrow$$

So that:

 $\llbracket \texttt{yield} \rrbracket = \texttt{d}(*) = \{(\sigma, \sigma)(\sigma', \sigma' \text{ return}) \text{ donem}\} \downarrow$ 

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# **Spawning Threads**

Recall:

$$\llbracket \operatorname{async} C \rrbracket = \operatorname{async}(\llbracket C \rrbracket^c)$$

Where:

-<sup>c</sup>: Proc → Pool is the extension to processes of the function
 -<sup>c</sup>: TSeq → PSeq of the same name from transition sequences to pure transition sequences which removes the marker return

• 
$$\operatorname{async}(P) =_{\operatorname{def}} \{(\sigma, \sigma \operatorname{return})u \mid \sigma \in \operatorname{Store}, u \in P\}$$

Note:  $\operatorname{async}(P^c)$  differs from d(P) only in the placement of the marker return: the former replaces it at the beginning.

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# **Denotational Semantics of Thread Pools**

- ▶: Pool<sup>2</sup> → Pool is the extension to thread pools of the binary function on pure transition sequences of the same name.
- Together with  $I =_{def} {done} \downarrow$  it forms a commutative monoid
- The semantics of a thread pool  $C_1, \ldots, C_n$  is given by:

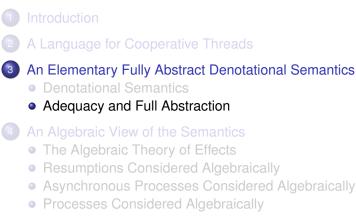
$$\llbracket C_1,\ldots,C_n \rrbracket = \llbracket C_1 \rrbracket^c \bowtie \ldots \bowtie \llbracket C_n \rrbracket^c \quad (n \ge 0)$$

- Note that [[ε]] = I
- Our domain of asynchronous processes AProc is the sub-ω-cpo of Pool none of whose elements contain done.
- We always have  $\llbracket C \rrbracket^c \in AProc.$

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# Adequacy Theorem for Pure Transition Sequences

Define:

$$\Gamma \Rightarrow \Gamma' \quad \text{iff} \quad \Gamma \longrightarrow_a^* \longrightarrow_c \Gamma'$$

and

$$\llbracket T, C \rrbracket = \operatorname{async}(\llbracket T \rrbracket) \circ \llbracket C \rrbracket$$

#### Theorem

The following are equivalent:

$$(\sigma_1, \sigma'_1) \dots (\sigma_n, \sigma'_n) \text{done} \in \llbracket T_1, C_1 \rrbracket^c \ (n > 0)$$

3 There are 
$$T_i, C_i, (i = 2, n)$$
 such that

• 
$$\langle \sigma_i, T_i, C_i \rangle \Rightarrow \langle \sigma'_i, T_{i+1}, C_{i+1} \rangle$$
, for  $1 \le i \le n-1$ 

• 
$$\langle \sigma_n, T_n, C_n \rangle \longrightarrow_a^* \langle \sigma'_n, \varepsilon, \text{skip} \rangle.$$

There is an analogous statement for  $(\sigma_1, \sigma'_1) \dots (\sigma_n, \sigma'_n) \in \llbracket T, C \rrbracket^c$ 

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## Adequacy Theorem for Runs

To account for uninterrupted running, we define, for  $P \in Pool$ :

 $\operatorname{runs}(P) = \{\sigma_1 \dots \sigma_n[\operatorname{done}] \mid (\sigma_1, \sigma_2)(\sigma_2, \sigma_3) \dots (\sigma_{n-1}, \sigma_n)[\operatorname{done}] \in P\}$ 

These runs are our observables.

#### Corollary

The following are equivalent:

2 There are 
$$T_i, C_i, (i = 2, n - 1)$$
 such that:  
 $\langle \sigma_1, T_1, C_1 \rangle \Rightarrow \ldots \Rightarrow \langle \sigma_{n-1}, T_{n-1}, C_{n-1} \rangle \longrightarrow_a^* \langle \sigma_n, \varepsilon, \text{skip} \rangle$ 

There is an analogous statement for  $\sigma_1 \dots \sigma_n \in \operatorname{runs}(\llbracket T_1, C_1 \rrbracket)$ 

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### Inequational Full Abstraction

#### Theorem

The following are equivalent, for any commands C and D:

- $\textcircled{O} \quad \llbracket C \rrbracket \subseteq \llbracket D \rrbracket$
- **②** For every context *C*, runs( $\llbracket C[C] \rrbracket^c$ ) ⊆ runs( $\llbracket C[D] \rrbracket^c$ ).

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The Algebraic Theory of Effects Resumptions Considered Algebraically Asynchronous Processes Considered Algebraically Processes Considered Algebraically



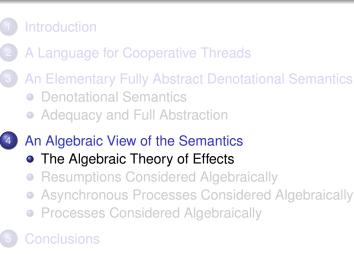
 Following Moggi we are interested in a monadic point of view, here using a continuous monad *T*(*P*) over ωCpo to model the set of computations for elements of *P*. We seek such a *T*<sub>Proc</sub> with:

$$Proc = T_{Proc}(1)$$

- To this end we seek a computationally interesting equational theory *L*<sub>Proc</sub> such that *T*<sub>Proc</sub> is the corresponding free algebra (better, free model) monad.
- This theory will be a variant of the theory for the classical resumptions monad and so we will also see how the trace model described above fits in with standard notions.

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## **Inequational Theories**

These are:

$$Th = (\Sigma, InEq)$$

where the operation arities  $f : n \rightarrow 1$  are given by  $\Sigma$  and InEq is a set of inequations

t ≤ u

over terms formed from these operation symbols.

One then has the usual notion of  $\Sigma$ -algebra in  $\omega$ Cpo and the free model—meaning modelling the inequations—monad over  $\omega$ Cpo is written  $T_{\text{Th}}$ .

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**Example** Nontermination: the theory  $L_{\Omega}$ 

 $\Omega \leq x$ 

Here  $T_{\Omega}$  is the usual lifting monad.

**Example** Hoare (Lower) Powerdomain: the theory  $L_H$ 

 $x \le x \cup y$   $y \le x \cup y$   $z \cup z \le z$ 

Here  $T_H$  is the lower powerdomain monad in  $\omega$ Cpo;  $T_H(P)$  is the free  $\omega$ -semilattice over P (meaning all countable sups) and it consists of all countably generated Scott closed sets.

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The Side-Effects Monad Considered Algebraically

Monad

$$T_{S}(P) = (\text{Store} \times P)^{\text{Store}}$$

#### Signature

lookup : Nat  $\rightarrow$  Vars update :  $1 \rightarrow$  Vars  $\times$  Nat

The corresponding generics are:

$$!: Vars \rightarrow Nat := : Vars \times Nat \rightarrow 1$$

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## Sample Equations for the Side-Effects Theory SE

update<sub>*l*,*v*</sub>(update<sub>*l'*,*v'*</sub>(*x*)) = update<sub>*l'*,*v'*</sub>(update<sub>*l*,*v*</sub>(*x*)) (if 
$$l \neq l'$$
)  
lookup<sub>*l*</sub>(... update<sub>*l*,*v*</sub>(*x*)...) = *x*

which last can be written in a finitary way as:

 $lookup_l((v : val).update_{l,v}(x)) = x$ 

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# Countably Infinitary Continuous Algebra

- Signature  $\Sigma = \{f : \overrightarrow{l_1}, \dots, \overrightarrow{l_m} \longrightarrow O_1, \dots, O_n\}$ , with  $\overrightarrow{l_1}, \dots, \overrightarrow{l_m}$  countably infinite sets, and  $O_1, \dots, O_n$  parameter spaces, being  $\omega$ -cpos, giving:
  - Function symbols  $f_{o_1,...,o_n}$  (for  $o_j \in O_j$ ), indexed by:

$$O =_{\text{def}} O_1 \times \ldots \times O_n$$

of arity

$$I =_{\mathrm{def}} (\prod \vec{I_1}) \times \ldots \times (\prod \vec{I_m})$$

Infinitary terms

$$f_{\overrightarrow{o}}\big(\langle t_{\overrightarrow{i_1},\ldots,\overrightarrow{i_m}}\rangle_{\overrightarrow{i_1},\ldots,\overrightarrow{i_m}}\big)$$

Note the indexed arguments.

 Inequations InEq consists of inequations t ≤ u between the (possibly) infinitary terms formed from the function symbols.

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## Models

 Algebras Carriers, being ω-cpos A, equipped with continuous maps

$$f_A: A^I \longrightarrow A^C$$

equivalently

$$f_A: O \times A^I \longrightarrow A$$

Models are such satisfying the inequations.

 Free Algebra Monad We obtain *T*<sub>Th</sub> giving the free such model; it is an ωCpo-monad.

**Remark** There is a useful finitary notation for such infinitary theories.

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## The Theory for Resumptions

We define:

$$L_{\text{Res}} = L_H \otimes ((L_S \otimes L_\Omega) + L_d)$$

where:

- $L_d$  is the theory of a unary operator, d, with no axioms.
- The axioms of *L* + *L*' are those of *L* and *L*' (we assume the operation symbols are disjoint).
- The axioms of L ⊗ L' are those of L + L' together with the commutativity of the operations of the one over the operations of the other (again assuming disjointness).

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# Basic *Q*-Transition Sequences

- Q a poset
- *Q*-transition:  $(\sigma, \sigma' x)$  where  $x \in Q$
- basic Q-transition sequence:  $(\sigma_1, \sigma_1), \ldots, (\sigma_n, \sigma_n)[(\sigma, \sigma' x)]$
- *Q*-BTrans is the partial order of *Q*-transition sequences where  $u \le v$  holds iff:

either 
$$u \leq_{p} v$$
  
or else  $\exists w, x \leq y. u \leq_{p} w(\sigma, \sigma' x) \land v = w(\sigma, \sigma' y)$ 

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Characterisation Theorem for Resumptions

#### Theorem

- Viewed as an  $L_{\text{Res}}$ -model,  $I_{\omega}(Q$ -BTrans) is  $T_{\text{Res}}(I_{\omega}^{\uparrow}(Q))$ .
- As a semilattice with a zero this is the solution in ωSL of the 'domain equation'

$${\sf R}\cong ({\sf S} imes ({\sf R}_{ot}+{\it Id}^{\uparrow}_{\omega}({\sf Q})))^{{\sf S}}$$

equivalently

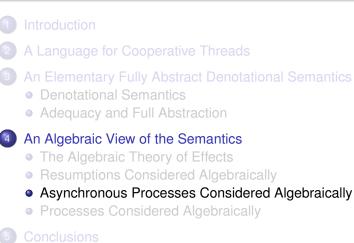
$$R \cong (S \times S) \times (R_{\perp} + Id_{\omega}^{\uparrow}(Q))$$

Q-BTrans is the solution in Pos of:

 $T \cong (S \times S) \times (T_{\perp} + Q)$ 

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### Asynchronous Processes

 $L_{AProc}$  is  $L_{Res}$  extended by a new constant halt with the axiom:

 $d(\Omega) \leq halt$ 

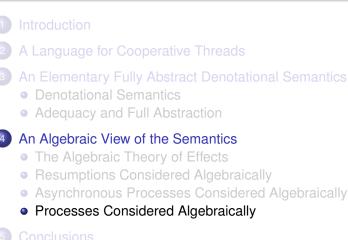
#### Theorem

- AProc is the initial  $L_{AProc}$ -model, i.e., it is  $T_{AProc}(0)$ .
- As a semilattice with a zero this is the solution in ωSL of the 'domain equation'

$$R \cong (S \times S) \times (R + 1)_{\perp}$$

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## The Theory for Processes

This is:

$$L_{\rm Proc} = L_{\rm Res} + L_{\rm Spawn}$$

where  $L_{\text{Spawn}}$  is the theory for **spawning** whose signature is that for  $L_{\text{Res}}$  together with two new operation symbols:

async :  $1 \longrightarrow AProc$ 

yield\_to :  $1 \longrightarrow AProc$ 

We write

 $P \cdot t$  for  $\operatorname{async}_P(t)$ 

 $t \cdot P$  for yield\_to<sub>P</sub>(t)

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#### Proc as a Proc-algebra

For  $P \in AProc$  and  $Q \in Proc$  we define:

$$P \cdot_{\text{Proc}} Q = \operatorname{async}(P) \circ Q$$
  
=  $\bigcup \{ (\sigma, \tau) u \bowtie w \mid u \in P, (\sigma, \tau) w \in Q \} \downarrow$ 

 $Q \cdot_{\text{Proc}} P = \bigcup \{ (\sigma, \sigma') u \bowtie v \mid (\sigma, \sigma') u \in P, v \in Q \} \downarrow$ 

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### First Group of Equations

These concern commutation with  $\cup$ :

$$(P \cup_{\text{AProc}} P') \cdot x = (P \cdot x) \cup (P' \cdot x)$$
$$P \cdot (x \cup y) = P \cdot x \cup P \cdot y$$
$$(x \cup y) \cdot P = x \cdot P \cup y \cdot P$$
$$x \cdot (P \cup_{\text{AProc}} P') = x \cdot P \cup x \cdot P'$$

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#### Equations for async

 $P \cdot \text{update}_{l,v}(x) = \text{update}_{l,v}(P \cdot x)$  $P \cdot \text{lookup}_{l}(\langle x_{v} \rangle_{v}) = \text{lookup}_{l}(\langle P \cdot x_{v} \rangle_{v})$  $P \cdot \Omega = \Omega$  $P \cdot d(x) = d(P \cdot x) \cup d(x \cdot P)$  $P \cdot (P' \cdot x) = (P \bowtie P') \cdot x$  $P \cdot (x \cdot P') = (P \cdot x) \cdot P' \cup (x \cdot P) \cdot P'$ 

(The last is redundant.)

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### Equations for yield\_to

 $x \cdot (\text{update}_{\text{AProc}})_{l,v}(P) = \text{update}_{l,v}(x \cdot P)$  $x \cdot (\text{lookup}_{\text{AProc}})_{l}(f) = \text{lookup}_{l}(\langle x \cdot f(v) \rangle_{v})$  $x \cdot \Omega_{\text{AProc}} = \Omega$  $x \cdot d_{\text{AProc}}(P) = d(x \cdot P) \cup d(P \cdot x)$  $x \cdot \text{halt}_{\text{AProc}} = d(x)$ 

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#### Transition Sequences for Processes

• *Q*-Trans 
$$=_{def}$$
 (*Q* × PSeq)-BTrans

Its elements have the form:

 $(\sigma_1, \sigma'_1) \dots (\sigma_m, \sigma'_m) [(\sigma, \sigma' \langle \mathbf{X}, (\overline{\sigma}_1, \overline{\sigma}'_1) \dots (\overline{\sigma}_n, \overline{\sigma}'_n) [\text{done}] \rangle)]$ 

The Algebraic Theory of Effects Resumptions Considered Algebraically Asynchronous Processes Considered Algebraically Processes Considered Algebraically

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**Characterisation Theorems for Processes** 

#### Theorem

- Viewed as an  $L_{\text{Proc}}$ -model,  $I_{\omega}(Q$ -Trans) is the free model over  $I_{\omega}^{\uparrow}(Q)$ .
- So as a Res-algebra:

$$T_{\operatorname{Proc}}(I_{\omega}^{\uparrow}(Q)) \cong T_{\operatorname{Res}}(\operatorname{Pool} \times I_{\omega}^{\uparrow}(Q))$$

Solution There is an isomorphism  $\theta: Q$ -Trans  $\rightarrow$  TSeq\{ $\varepsilon$ }, where  $Q = \{$ return $\}$  and so, as a Proc-algebra:

$$\operatorname{Proc} \cong \mathcal{I}_{\omega}(\operatorname{TSeq} \setminus \{\varepsilon\}) \cong \mathcal{T}_{\operatorname{Proc}}(1)$$

# Some Algebraic Reflections

- This is applied domain theory where one is interested in particular models and, particularly, their algebraic structure.
- Having free algebras is a condition on a domain theory: cf. Martin Hyland's 'reasons for domain theory' Part 1.
- Here, some structure, particularly the semilattice structure, is 'nice' mathematically; the actions are less so.
- Still, parallel constructs are typically not even algebraic operations.
- In the Proc characterisation theorem, part 2, we do not get the correct left action structure, though there is a wrong structure as Pool is a (commutative) monoid.
- Perhaps a Hopf shuffle algebra would help for a 'rational algebraic analysis' (cf. Martin Hyland's categorical rational reconstructions in domain theory).

# Possible Future Work

- Must semantics (compact sets of transition sequences)
- Add variable declaration: a challenge, at the least, for the algebraic part.
- Add higher-types. Can do as have monad, but full abstraction is another matter.
- Change notion of observations: runs with stuttering or mumbling.
- Fairness: all threads in the pool will eventually be chosen in any infinite run.
- Lower level semantics, with block treated as an exception causing a rollback; can then do *C* **orelse** *C*'.
- What equations hold not involving side-effects, conditionals or while loops? Example:

 $\llbracket (\texttt{async} (C; \texttt{async} (D)) \rrbracket = \llbracket \texttt{async} (C; \texttt{yield}; D) \rrbracket$