Semilattices and Domains

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Abstract. As everyone knows, one popular notion of ***Scott domain*** is defined as a bounded complete algebraic cpo. These are closely related to ***algebraic lattices*:** (*i*) A Scott domain becomes an algebraic lattice with the adjunction of an (isolated) top element. (*ii*) Every non-empty Scott-closed subset of an algebraic lattice is a Scott domain. Moreover, the isolated (= compact) elements of an algebraic lattice form a semilattice (under join). This semilattice has a zero element, and, provided the top element is isolated, it also has a unit element. The algebraic lattice itself may be regarded as the ideal completion of the semilattice of its isolated elements. (***Comment:*** The author apologizes for using the adjective "Scott" so often. But, remember, he did not invent the terminology!)

Section 1. A universal domain. Let

$$\mathfrak{A} == \langle A, 0, 1, \vee \rangle$$

be a (join) semilattice (with unit and $0 \neq 1$). Let $\leq$ be the partial ordering of the semilattice $\mathfrak{A}$ defined as usual by $a \leq b \Longleftrightarrow a \vee b == b$. We denote by $\|\mathfrak{A}\|$ the ***ideal completion*** (without necessarily a top) as being the set of ***proper ideals***:

$$\{X \subseteq A \mid 0 \in X \ \& \ 1 \notin X \ \& \ \forall\, a, b \in A\,[\, a, b \in X \Longleftrightarrow a \vee b \in X\,]\,\}.$$

Under set inclusion, $\|\mathfrak{A}\|$ becomes a Scott domain.

Note that in case $\forall\, a, b \in A\,[\, a \vee b == 1 \Longrightarrow a == 1 \text{ or } b == 1\,]$ holds in the semilattice, then the completion $\|\mathfrak{A}\|$ is an algebraic lattice with a top element. (***Why?***) As remarked in the abstract, the following result is well known:

Theorem. *Up to isomorphism, every Scott domain can be obtained in this way.*

Next, let $\mathfrak{P} == \langle P, 0, 1, \vee \rangle$ be the semilattice part of the ***free Boolean algebra*** on denumerably many generators (*i.e.*, the Boolean algebra of ***classical propositional calculus***). As is also well known, the Stone space of $\mathfrak{P}$ (regarded as a Boolean algebra) is (homeomorphic to) the Cantor set (as a subset of the real unit interval). The standard result of Stone Duality implies:

Theorem. *The Scott domain* $\|\mathfrak{P}\|$ *is isomorphic to the domain of* **open subsets** *of the Cantor set — with the compact, whole Cantor set* **removed**.

Not as well known is the:

Theorem. $\|\mathfrak{P}\|$ *is a* **universal** *Scott domain for the countably based Scott domains.*

The universality can be proved as follows. We need to know that $\mathfrak{P}$, regarded as a Boolean algebra, contains an isomorphic copy of every countable Boolean algebra as a ***subalgebra***. This is a consequence of the fact that a countably generated Boolean algebra is the union (direct limit) of a chain of ***finite*** subalgebras. Inasmuch as a finite Boolean algebra is atomic, say, with n atoms, it can be embedded in $\mathfrak{P}$ by taking n pairwise, non-zero elements of $\mathfrak{P}$ to match the atoms. Now, any finite extension of the finite Boolean algebra just ***subdivides*** the atoms of the smaller algebra into disjoint atoms of the larger algebra. Because $\mathfrak{P}$ has infinitely many independent generators, the embedded copy of the smaller finite algebra can easily have the images of its atoms in $\mathfrak{P}$ similarly subdivided into disjoint parts. In this way, any embedding can be ***extended*** to an embedding of a superalgebra. By iterating these extensions, the whole countable algebra can be isomorphically embedded in $\mathfrak{P}$. And an easy corollary is that $\mathfrak{P}$, regarded as a semilattice, contains an isomorphic copy of every countable semilattice.

To see this, all we have to do is embed a ***countable*** semilattice

$$\mathfrak{A} == \langle A, 0, 1, \vee \rangle$$

into a ***countable*** Boolean algebra. Thus, consider the Boolean algebra $\mathcal{P}(A \setminus \{1\})$ of all subsets of A not containing the element 1. Define a mapping $\mu : A \to \mathcal{P}(A \setminus \{1\})$ by:

$$\mu(a) = \{b \in A \setminus \{1\} \mid a \nleq b\}.$$

It is easy to see that this is a semilattice embedding. The range of μ generates a countable Boolean algebra, which can be embedded into our universal $\mathfrak{P}$. Hence, $\mathfrak{A}$ has a semilattice embedding into $\mathfrak{P}$ as well.

Thus, from now on, restricting attention to countable semilattices, we regard various semilattices $\mathfrak{A}$ as just being ***subsemilattices*** of the fixed universal semilattice $\mathfrak{P}$.

Note, too, that as an algebraic structure, and as regarded as a Boolean algebra, $\mathfrak{P}$ can be ***enumerated*** by suitably chosen Gödel numbers in such a way that all the Boolean operations and the partial ordering are ***(primitive) recursive***. Moreover, given any recursive semilattice, the above proof can be used to show that it can be given a ***recursive embedding*** onto a recursive subsemilattice of $\mathfrak{P}$.

Section 2. The lattice of subsemilattices. Let $\mathbb{S}$ be the collection of all subsemilattices of $\mathfrak{P}$ regarded just as a collection of subsets of P. Under inclusion, as is well known, $\mathbb{S}$ is an ***algebraic lattice***. The bottom element of $\mathbb{S}$ is $\{0, 1\}$, and the top element is P, which, by the way, is not isolated. (***Why?***) In fact, the isolated elements of $\mathbb{S}$ are just the finite subsemilattices of $\mathfrak{P}$, and every finite subset of P generates a finite subsemilattice.

Let $\mathbb{F}$ denote the semilattice of finite elements of $\mathbb{S}$ together with a top element (actually it could be P itself). It follows from the remarks of the preceding paragraph that the semilattice $\mathbb{F}$ can be given a recursive embedding into $\mathfrak{P}$ and is indeed isomorphic to a recursive element of $\mathbb{S}$.

For $A \in \mathbb{S}$, let us now slightly modify the definition of $||\mathfrak{A}||$ in order to make some comparisons easier. Use for $p \in P$ the notation $\downarrow p == \{q \in P \mid q \leq p\}$. And for sets, also write $\downarrow X == \{q \in P \mid \exists\, p \in X . q \leq p\}$.

We then define $||A|| == \{\downarrow (X \cap A) \mid X \in ||\mathfrak{P}||\}$. With this notation $||A||$ is a subdomain of $||P|| == ||\mathfrak{P}||$. Indeed, the mapping $X \mapsto \downarrow (X \cap A)$ is a ***continuous finitary projection*** of $||P||$ onto $||A||$. And note that $||A|| \cup \{P\}$ is a lattice under the join operation, which can be defined as:

$$X \vee Y == \{p \vee q \mid p \in X \;\&\; q \in Y\}.$$

What good is all this? Well, some years ago the author and, independently, Glynn Winskel introduced the notion of ***information systems*** for constructing Scott domains. More recently Winskel in his excellent textbook, *The Formal Semantics of Programming Languages: An Introduction* (MIT Press, 1993), devotes Chapter 12 to this theory in order to show how to solve recursive domain equations. In lectures at UC Berkeley this spring the author realized that the all the necessary structure of information systems can be explained just by using semilattices in what he considers to be a very elementary way.

To understand how semilattices can be used in this way, interpret (informally) the element $0 \in P$ to stand for "no information" and the element $1 \in P$ to stand for "too much information". Note that ***all*** the semilattices $A \in \mathbb{S}$ use the ***same*** elements 0 and 1 in this way. The relationship $q \leq p$ then means that q has "less information" than p. Again, ***all*** the semilattices $A \in \mathbb{S}$ use the ***same*** partial ordering coming from P.

Consider next two elements $p, q \in P$. How should we interpret the element $p \vee q$? The answer is to "join" the information in p to the information in q. However, were it to turn out that $p \vee q == 1$, then we need to call p "***informationally inconsistent***" with q.

Note that we insisted for ideals $X \in \|P\|$ that $1 \notin X$. This means that all our ideal elements are "***consistent***". Note, too, that one notion of join and one notion of consistency work for ***all*** the semilattices in $\mathbb{S}$.

And, one hopes for making life seem a little simpler, the only difference between ideals $X \in \|A\|$ and other ideals in $Y \in \|P\|$ is that these ideals are "generated" by information confined to A. Indeed, for each $X \in \|A\|$ we have $X == \downarrow(X \cap A)$.

Section 3. Constructing domains. We next have to ask an important question.

How do semilattices help in defining domain constructs? Let us examine first the constructions of ***products*** and ***sums***. The key idea is that, since the isolated elements of a Scott domain determine the whole domain, one tries to define the construct on the isolated elements. For products and sums this proves to be quite easy.

Because P is being used as a universal domain, we need a kind of general ordered pair in the semilattice P. A convenient one can be obtained with properties as follows:

Theorem. *There is a recursive operation $\langle\langle p, q\rangle\rangle$ on P such that:*

(*i*) $\langle\langle 0, 0\rangle\rangle == 0$;

(*ii*) $\langle\langle p, q\rangle\rangle == 1 \Longleftrightarrow p == 1$ or $q == 1$;

(*iii*) $\langle\langle p_0, q_0\rangle\rangle \vee \langle\langle p_1, q_1\rangle\rangle == \langle\langle p_0 \vee q_1, p_0 \vee q_1\rangle\rangle$; *and*

(*iv*) $\langle\langle p_0, q_0\rangle\rangle \leq \langle\langle p_1, q_1\rangle\rangle \Longleftrightarrow p_1 == 1$ or $q_1 == 1$ or $[\, p_0 \leq p_1 \;\&\; q_0 \leq q_1 \,]$.

Moreover, the range of the operation is a recursive subset of P.

Before indicating a proof, we introduce three definitions.

Definition. For $X,\ Y \subseteq P$, let $X \times Y == \{\, \langle\langle p, q\rangle\rangle \mid p \in X \;\&\; q \in Y \,\}$.

Definition. $H == \{\, \langle\langle p, q\rangle\rangle \mid [p == 0 \;\&\; q == 0]$ or $[p \neq 0 \;\&\; q \neq 0] \,\}$.

Definition. For $A,\ B \in \mathbb{S}$, let $A \times_s B == \{\, \langle\langle p, q\rangle\rangle \mid p \in A \;\&\; q \in B \,\} \cap H$.

From the theorem above about $\langle\langle p, q\rangle\rangle$ it follows that whenever $A, B \in \mathbb{S}$, then $A \times B \in \mathbb{S}$ as well. (Clauses (*i*) and (*ii*) of the theorem come into play since all the subsemilattices of P have to use the same 0 and 1.) Additionally this product definition is a (computable and) continuous operation on the Scott domain $\mathbb{S}$.

Now, when we look at the elements of $\|A \times B\|$, we find each $Z \in \|A \times B\|$ corresponds uniquely to the pair of elements $X \in \|A\|$ and $Y \in \|B\|$, where $X == \{p \mid \langle\langle p, 0\rangle\rangle \in Z\}$ and $Y == \{q \mid \langle\langle 0, q\rangle\rangle \in Z\}$. Moreover, for any two $X \in \|A\|$ and $Y \in \|B\|$, we find that $Z == \downarrow(X \times Y) \in \|A \times B\|$. Filling in some minor details here, we can then prove:

Theorem. *$\|A \times B\|$ is indeed isomorphic to the **domain product** of $\|A\|$ and $\|B\|$.*

It is easy to see that $H \in \mathbb{S}$. A simple proof shows that $\|A \times_s B\|$ is isomorphic to the familiar ***smash product*** of $\|A\|$ and $\|B\|$. And, in terms of semilattices we again have a (computable and) continuous operation on the Scott domain $\mathbb{S}$ with a simple definition.

We now have to return to defining pairs and to proving the theorem. So, we again think of P as a free Boolean algebra, say on generators $v_0, v_1, \ldots, v_n, \ldots$. Two homomorphisms of P can be defined .

Let $\epsilon_0 : P \to P$ be the (recursive) Boolean monomorphism where $\epsilon_0(v_n) == v_{2n}$.

Let $\epsilon_1 : P \to P$ be that other (recursive) Boolean monomorphism where $\epsilon_1(v_n) == v_{2n+1}$.

Then, define $\langle\langle p, q\rangle\rangle == \epsilon_0(p) \vee \epsilon_1(q)$.

There is only one part of one clause of the theorem that needs a little thought to prove. Thus, assume $\langle\langle p_0, q_0\rangle\rangle \leq \langle\langle p_1, q_1\rangle\rangle$ and both $p_1 \neq 1$ and $q_1 \neq 1$. In view of the last assumption, there is a Boolean homomorphism $h : P \to P$ such that $h(\epsilon_1(q_1)) == 0$, but $h(\epsilon_0(p)) == p$, for all $p \in P$.

This would imply that $p_0 \leq p_0 \vee h(\epsilon_1(q_0)) \leq p_1$. By a similar argument we can prove under the same assumptions that $q_0 \leq q_1$.

We note next that the pairing operation on P can be used for other constructs on semilattices.

Definition. For $A_0, A_1, \ldots, A_m \in \mathbb{S}$, the ***separated sum*** is defined as:

$$A_0 + A_1 + \ldots + A_m ==$$
$$\{0\} \cup \{\langle\langle p, v_0\rangle\rangle \mid p \in A_0\} \cup \{\langle\langle p, \neg v_0 \vee v_1\rangle\rangle \mid p \in A_1\} \cup \ldots \cup \{\langle\langle p, \neg v_0 \vee \neg v_1 \vee \ldots \vee \neg v_{m-1} \vee v_m\rangle\rangle \mid p \in A_m\}$$

Definition. For $A_0, A_1, \ldots, A_m \in \mathbb{S}$, the ***coalesced sum*** is defined as:

$$A_0 +_c A_1 +_c \ldots +_c A_m == (A_0 + A_1 + \ldots + A_m) \cap H.$$

We see that both sums produce semigroups from semigroups, and both are continuous and computable operations on $\mathbb{S}$. Both operations can also easily be extended to ***denumerably many*** terms.

Definition. For $A \in \mathbb{S}$, the ***lift*** is defined as: $A_\perp == \{0\} \cup \{\langle\langle p, 0\rangle\rangle \vee v_1 \mid p \in A\}$.

Definition. For $A \in \mathbb{S}$, the ***drop*** is defined as: $A^\top == \{1\} \cup \{\langle\langle p, 0\rangle\rangle \wedge v_1 \mid p \in A\}$.

We see that lifts and drops produce semigroups from semigroups, and they are continuous and computable operations on $\mathbb{S}$. We have to leave to the reader the task of checking that these operations on semilattices produce the desired results for obtaining these Scott domains:

$$\|A_0 + A_1 + \ldots + A_m\|, \ \|A_0 +_c A_1 +_c \ldots +_c A_m\|, \ \|A_\perp\|, \text{ and } \left\|A^\top\right\|$$

with the right domain properties. But we hope experience with standard Domain Theory makes this obvious.

Section 4. Function spaces as domain. And now we come to the next question.

Is it just as easy to construct the semilattices corresponding to function spaces? The answer is "not quite". The author did not find any Boolean construction similar to the definition of $\langle\langle p, q\rangle\rangle$ to give the isolated elements of the space of continuous mappings from $\|P\|$ into $\|P\|$. Instead, the required semilattice has to be constructed ***formally*** as a countable semilattice and then embedded into P. Here is the final result of the embedding.

Theorem. *There is a recursive operation* $(p \Rightarrow q)$ *on* P — *defined when* $p \neq 1$ — *such that:*

(*i*) $(p \Rightarrow 1) == 1$;

(*ii*) $\bigvee_{i<k} (p_i \Rightarrow q_i) == 1 \implies \exists\, r \neq 1. \bigvee \{q_i \mid p_i \leq r\} == 1$; and

(*iii*) $(r \Rightarrow s) \leq \bigvee_{i<k} (p_i \Rightarrow q_i) \iff \bigvee_{i<k} (p_i \Rightarrow q_i) == 1$ or $s \leq \bigvee \{q_i \mid p_i \leq r\}$.

Moreover, the semilattice generated by the range of the operation is a recursive subset of P.

Suppose we have such an operation. There are several properties that follow.

Lemma. *Keeping in mind that, in using the operation* $(\bullet \Rightarrow \bullet)$*, we always assume the first argument is* ***not*** *1, then the following properties hold:*

(*i*) $(p \Rightarrow q) == 0 \Longleftrightarrow q == 0$;

(*ii*) $\bigvee_{i<k} (p_i \Rightarrow q_i) == 1 \Longleftrightarrow \exists\, r \neq 1. \bigvee \{q_i \mid p_i \leq r\} == 1$;

(*iii*) $(p \Rightarrow q) == 1 \Longleftrightarrow q == 1$;

(*iv*) $p_0 \leq p_1 \neq 1 \;\&\; q_1 \leq q_0 \Longrightarrow (p_1 \Rightarrow q_1) \leq (p_0 \Rightarrow q_0)$; *and*

(*v*) $\bigvee_{i<k} (p \Rightarrow q_i) == \left(p \Rightarrow \bigvee_{i<k} q_i\right)$.

Definition. For $A, B \in \mathbb{S}$, the ***normal function space basis*** is defined as:

$$(A \Rightarrow B) == \left\{ \bigvee_{i<k} (p_i \Rightarrow q_i) \;\middle|\; \forall\, i < k\,[\, p_i \in A \setminus \{1\} \;\&\; q_i \in B\,]\right\}.$$

Definition. For $A, B \in \mathbb{S}$, the ***strict function space basis*** is defined as:

$$(A \Rightarrow_s B) == \{0, 1\} \bigcup \left\{ \bigvee_{i<k} (p_i \Rightarrow q_i) \;\middle|\; \forall\, i < k\,[\, p_i \in A \setminus \{0, 1\} \;\&\; q_i \in B \setminus \{0, 1\}\,]\right\}.$$

On the basis of these definition, for $A, B \in \mathbb{S}$, it is clear that both $(A \Rightarrow B)$ and $(A \Rightarrow_s B)$ are in $\mathbb{S}$ and that these are continuous operations.

In order to understand the connections with function spaces, take any continuous function $\Phi : \|A\| \to \|B\|$, where $A, B \in \mathbb{S}$ are given. Now, a continuous function is completely determined by its action on ***isolated*** elements, which, in the case of the first space are the $\downarrow p \in \|A\|$. And these are in a one-one correspondence with the $p \in A \setminus \{1\}$. Let $|\Phi|$ be the ideal in $\|A \Rightarrow B\|$ generated by the set

$$\{(p \Rightarrow q) \mid q \in \Phi(\downarrow p) \bigcap B \;\&\; p \in A \setminus \{1\}\}.$$

From the properties of the operation $(\bullet \Rightarrow \bullet)$ mentioned above, we then check that for $p \in A \setminus \{1\}$ and $q \in B$ we have $(p \Rightarrow q) \in |\Phi| \Longleftrightarrow q \in \Phi(\downarrow p)$. This shows that the functions Φ and the ideals of the form $|\Phi|$ are in a one-one correspondence.

To show that every element of $\|A \Rightarrow B\|$ is of the proper form, suppose $F \in \|A \Rightarrow B\|$. Consider the function $\Phi_F : \|A\| \to \|B\|$ defined as

$$\Phi_F(X) == \{q \mid \exists\, (p \Rightarrow q) \in F.\, p \in X\} \text{ for } X \in \|A\|.$$

After showing that this function is well defined, it is obvious that it is continuous. Moreover, $|\Phi_F| == F$.

Section 5. Recursive domain equations. Inasmuch as $\mathbb{S}$ is an effectively presented algebraic lattice, and because constructors for domains can be given by easy-to-define continuous (and computable) mappings from $\mathbb{S}$ into itself, the standard Fixed-Point Theorem for recursive definitions on domains can be applied directly to give ***recursive constructions of domains***.

Indeed, a quick look at the definitions of $A \times B$, $A \times_s B$, $A + B$, $A +_c B$, $A_\bot$, $A^\top$, $A \Rightarrow B$, and $A \Rightarrow_s B$ will show that each of these operation take elements of $\mathbb{S}$ to elements of $\mathbb{S}$, and, moreover, $\mathbb{F}$, the semilattice of finite elements of $\mathbb{S}$, is also closed under these operation. Furthermore, on $\mathbb{S}$ the operations are continuous, and on $\mathbb{F}$ they are computable. The same can be said of any composition of these operations.

Do we not now have models for the λ-calculus constructed from semilattices? Of course! There are many. We could start with any $A \in \mathbb{S}$ and solve the domain equation $D == A +_c (D \Rightarrow D)$ to obtain a $D \in \mathbb{S}$. Then $||D||$ can be regarded as a λ-calculus model. There are many other (well known) uses of domain equations also leading to models. Note, too, that $||P||$ is itself a model, inasmuch as $||P \Rightarrow P||$ is at once seen as a continuous retraction of $||P||$.

Section 6. Some relations to the literature. The author pointed out early on in the development of Domain Theory the usefulness of universal domains of various kinds, and that approach was elaborated by many other authors—also for categories beyond that of Scott domains considered here.

The universal domain for countably based algebraic ***lattices*** is Pω, the powerset of the integers, which is "simpler" than the Scott domain $||P||$ used here. As a λ-calculus model, this is the ***graph model***, originally proposed by Plotkin and analyzed extensively in:

G.D. Plotkin, *Set-theoretical and other elementary models of the λ-calculus*, **Theoretical Computer Science**, vol. 121 (1993), pp. 351-409.

Plotkin also discusses the so-called filter λ-models and Engeler-style models. Some early references to the works of the "Torino School" are:

M. Coppo, F. Honsell, M. Dezani-Ciancaglini and G. Longo, *Extended type structures and filter λ-models*, **Logic Colloquium '82**, North-Holland, Amsterdam 1984, pp. 241-262.

H. Barendregt, M. Coppo and M. Dezani-Ciancaglini, *A filter lambda model and the completeness of type assignment*, **The Journal of Symbolic Logic**, vol. 48 (1983), pp. 931-940.

M. Coppo, *Completeness of type assignment in continuous lambda models,* **Theoretical Computer Science**, vol. 29 (1984), pp. 309-324.

Further references can be found on Mario Coppo's home page. In the last cited paper, he writes in his abstract:

- The completeness of Curry's rules for assigning type schemes to terms of the pure lambda-calculus has been proved by Hindley (1983) and Barendregt et al. (1983) using models of syntactic nature. A first result of this paper is a completeness proof with respect to the model Pω (as asked by Scott (1976)). Moreover, an extension of Curry's system in which type schemes can be assigned to the fixed point combinator is introduced, together with a notion of type semantics for which it is proved sound and complete (answering a question of Scott (1980)). Also in this case, completeness is proved with respect to the model Pω. All results also hold for the alternative notions of type semantics proposed by Hindley (1983) and Scott (1976, 1980).

Indeed, the filter λ-models were originally invented to give a semantics to type assignments, and the work of Coppo and collaborators ties together many results in this vein. The author wishes to stress here, however, that the universal domain $||P||$ used here has a much broader motivation. It contains not only the data for the function-space construction, but—as shown above—ways of constructing a large selection of other structures. The rules for the "step functions" in Section 4 have been noted by many authors, but they were the key insights for the authors original realization that the category of countably based Scott domains is a cartesian closed category. Further investigations of λ-calculus model constructions have been presented in this recent paper:

M. Hyland, M. Nagayama, J. Power, and G. Rosolini, *A category theoretic formulation for Engeler-style models of the untyped λ-calculus*, **Electronic Notes in Theoretical Computer Science**, vol. 161 (2006), pp. 43-57.

They make this comment at the end of their paper:

- We close by explaining the programme of work which we initiate here. Filter lambda models as introduced by the Torino school [...] are usually taken to amount to a presentation of domain theoretic models. Certainly they are lambda models as they appear to come from categories with enough points. But our analysis of the Engeler model is that it naturally arises from a category without enough points. So in our formulation it is naturally a lambda algebra in the established terminology. Now the Engeler model is taken to be part of the general family of filter models. So this raises (at least for us) questions along the following lines. Which filter models really are domain models (after all nobody doubts Scott's D_∞), and which are naturally something else? There seems much more to understand about concrete constructions of models for the lambda calculus (that is, in general about lambda algebras).

It is doubtful that the present paper contributes something new to their programme, but the author at least hopes the discussion here shows that certain model constructions can be more easily explained than had been realized before.