The limit-colimit coincidence theorem for $\mathcal{Q}$-categories

MATEUSZ KOSTANEK and PAWEŁ WASZKIEWICZ [†]

Theoretical Computer Science, Jagiellonian University, ul. S. Łojasiewicza 6, 30-348 Kraków.

We prove that the category of $\mathcal{I}$-cocomplete separated $\mathcal{Q}$-categories has bilimits of expanding sequences. This result generalises on various levels the well-known theorem of domain theory that guarantees existence of bilimits in the category of directed-complete posets and Scott-continuous maps.

1. Motivation and related work

In 1973 William Lawvere published a paper (reprinted as (Lawvere 2002)) where he explained that partial orders and metric spaces are examples of categories enriched in a closed category. Indeed, preorders are categories enriched over the two-element Boolean algebra **2**, while (generalized) metric spaces are categories enriched over $([0, \infty], +)$. Lawvere's idea has been extremely influential in the forthcoming years. For example it led to the development of a unified categorical/algebraic description of many familiar elementary structures in mathematics: topology, uniformity, order, metric, etc. (Clementino and Hofmann 2003; Clementino and Tholen 2003; Clementino *et al* 2004; Hofmann 2007). At the same time, the idea was taken up by computer scientists in the hope that since posets have been so successfully used to create denotational semantics of programming languages, generalized metric spaces (gmses) could be useful too, especially for expressing quantitative properties of programs. As a consequence there are many studies of gmses concerned not as much with generalizing metric spaces as with generalizing domains and *domain theory*: (Rutten 1996; Flagg and Kopperman 1997; Flagg 1997) speak about *Alexandroff and Scott topologies* for generalized metric spaces; (America and Rutten 1989; Wagner 1994; Flagg and Kopperman 1995) are devoted to solving *recursive domain equations* in gmses; (Bonsangue *et al* 1998) proposes *powerdomains* for gmses; (Vickers 2005) completes a gms using rounded filters of *formal balls*, (Zhang and Fan 2005; Waszkiewicz 2009) analyze *approximation* and *continuity* in $\mathcal{Q}$-posets, etc. Our work contributes to the same line of research: in this paper we present a generalisation of the limit-colimit coincidence theorem from domain theory, which states, in the nomenclature of (Abramsky and Jung 1994), that the category **Dcpo** has bilimits of expanding sequences. Concretely, we

prove that the category of $\mathcal{I}$-cocomplete separated $\mathcal{Q}$-categories has bilimits of expanding sequences.

One of the prominent features of our paper is that we work with $\mathcal{Q}$-categories that are cocomplete relative to a distinguished class of $\mathcal{Q}$-ideals $\mathcal{I}$. The idea of cocompleteness relative to a class $\mathcal{I}$ of weights was developed in (Kelly 1982; Albert and Kelly 1998; Kock 1995; Kelly and Schmitt 2005; Kelly and Lack 2000; Schmitt 2006), and well-explained in Lai and Zhang (Lai and Zhang 2007). Our approach is justified by the observation that there is no canonical choice for $\mathcal{Q}$-categorical counterparts of order-ideals. They can be generalized to several non-equivalent concepts on the $\mathcal{Q}$-level (e.g. forward Cauchy nets, flat modules, FSW-ideals) which nevertheless yield the same definition in both the metric and the order-theoretic case (Flagg *et al* 1996; Bonsangue *et al* 1998; Schmitt 2006; Vickers 2005). Therefore, one obtains different notions of cocompleteness for $\mathcal{Q}$-categories based on a specific choice of $\mathcal{Q}$-ideals. The starting point of our paper is the conviction that one does not have to make this choice right at the beginning, and — as a consequence — we introduce $\mathcal{Q}$-categories that are $\mathcal{I}$-cocomplete (see Section 2.9). Accordingly, the limit-colimit coincidence theorem that we prove in Section 3 is relative to both the choice of the quantale and the choice of $\mathcal{Q}$-ideals.

Finally, we should mention that our exposition, while categorical, is kept close to domain-theoretic language of (Abramsky and Jung 1994; Gierz *et al* 2003). In fact, we assume familiarity with domain theory and we most often motivate $\mathcal{Q}$-categorical constructions with order-theoretic examples.

Acknowledgements

2. Preliminaries

In this section we introduce all categories that will be studied in this paper. Our exposition is intended to highlight a way these categories emerge from generalizing order theory and domain theory. For other introductions to the subject we encourage the reader to consult for example (Hofmann 2007) or (Waszkiewicz 2009).

2.1. *Quantales*

In this paper $\mathcal{Q} = (Q, \leqslant, \otimes, \mathbf{1})$ is a certain unital quantale (see e.g. (Rosenthal 1990)), i.e.: (a) $(Q, \leqslant)$ is a complete lattice; (b) Q is equipped with an associative, commutative operation $\otimes \colon Q \times Q \to Q$ called tensor that preserves arbitrary suprema, i.e. $a \otimes \bigvee S = \bigvee \{a \otimes s \mid s \in S\}$ for all $a \in Q$ and $S \subseteq Q$; (c) Q has a distinguished element $\mathbf{1}$, called unit, that satisfies $a \otimes \mathbf{1} = a$ for all $a \in Q$; (d) $\mathbf{1}$ is the top element of Q.

The right adjoint to tensor is defined via: $a \otimes b \leqslant c$ iff $a \leqslant b \multimap c$, for all $a, b, c \in Q$. The

least element of Q is denoted by $\perp$.

Examples of quantales that satisfy (a)-(d):

Example 2.1. The two element lattice $\{\perp, \mathbf{1}\}$ with $\perp \leqslant \mathbf{1}$, and with $\otimes$ being infimum with respect to $\leqslant$. It is denoted by $\mathbf{2}$.

Example 2.2. The unit interval $[0, 1]$ in the order opposite to the natural one ($\leqslant$), with the truncated addition as $\otimes$:

$$a \otimes b = \begin{cases} a + b & \text{if } a + b \leqslant 1, \\ 1 & \text{otherwise} \end{cases}.$$

Here $\multimap$ is the opposite to the truncated substraction.

Example 2.3. The lattice $[0, \infty]$ in the order opposite to the natural one, with addition, is a quantale such that $\multimap$ is the opposite to the truncated substraction.

Example 2.4. Any complete Heyting algebra with infimum as tensor and top element as unit.

2.2. $\mathcal{Q}$-relations

The category $\mathcal{Q}$-**Rel** has sets as objects, and functions $r \colon X \times Y \to \mathcal{Q}$ as morphism; a morphism $r \colon X \times Y \to \mathcal{Q}$ will be conveniently denoted as $r \colon X \longrightarrow Y$, so we can write a composition of $r \colon X \longrightarrow Y$ with $s \colon Y \longrightarrow Z$ as:

$$X \xrightarrow{\ r\ } Y \xrightarrow{\ s\ } Z$$

with

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z). \tag{1}$$

The identities in $\mathcal{Q}$-**Rel** are $\mathcal{Q}$-relations defined for each set X as $1_X \colon X \longrightarrow X$:

$$1_X(x, y) = \begin{cases} \mathbf{1} & \text{if } x = y, \\ \perp & \text{otherwise.} \end{cases}$$

There is a functor $\mathbf{Set} \to \mathcal{Q}$-**Rel** which maps objects identically and interprets a map $f \colon X \to Y$ as a $\mathcal{Q}$-relation $f \colon X \longrightarrow Y$:

$$f(x, y) = \begin{cases} \mathbf{1} & \text{if } f(x) = y, \\ \perp & \text{otherwise.} \end{cases} \tag{2}$$

The hom-sets of $\mathcal{Q}$-**Rel** carry the pointwise order of $\mathcal{Q}$, so that $\mathcal{Q}$-**Rel** becomes an ordered category. In fact, each hom-set of $\mathcal{Q}$-**Rel** is a complete lattice.

The category $\mathcal{Q}$-**Rel** has a contravariant involution $(\mathcal{Q}\text{-}\mathbf{Rel})^{\mathrm{op}} \to \mathcal{Q}$-**Rel** which maps objects identically and assigns to $r \colon X \longrightarrow Y$ its opposite relation $r^\circ \colon Y \longrightarrow X$. When

applied to a map f, one obtains an adjunction $f \dashv f^{\circ}$.

Recall that $g \dashv h$ in $\mathcal{Q}\text{-}\mathbf{Rel}$ means that $g \cdot h \leqslant 1$ and $h \cdot g \geqslant 1$.

2.3. $\mathcal{Q}$-categories

$\mathcal{Q}$-categories are structures that we can present as generalizations of preorders. Note that a relation (i.e. a **2**-relation) $X \colon X \longmapsto X$ is a preorder iff $1_X \subseteq X$ and $X \cdot X \subseteq X$. Analogously, a $\mathcal{Q}$-*category* is a set X with a $\mathcal{Q}$-relation $X \colon X \longmapsto X$ satisfying:

(r) $1_X \leqslant X$ (reflexivity),
(t) $X \cdot X \leqslant X$ (transitivity),

or, pointwise:

(r) $\mathbf{1} \leqslant X(x, x)$, for all $x \in X$,
(t) $X(x, y) \otimes X(y, z) \leqslant X(x, z)$, for all $x, y, z \in X$.

Let us now extend the analogy with preorders further: a $\mathcal{Q}$-category X is *separated* if in addition to **(r)** and **(t)** above, it satisfies:

(a) $\mathbf{1} \leqslant X(x, y)$ and $\mathbf{1} \leqslant X(y, x)$ imply $x = y$ for all $x, y \in X$.

For the purpose of this paper separated $\mathcal{Q}$-categories will be called $\mathcal{Q}$-*posets*.

The $\mathcal{Q}$-relation $X \colon X \longmapsto X$ that we met in the definition of a $\mathcal{Q}$-category is often called the *structure* of X, e.g. in (Hofmann 2007), or a distance on X (Flagg *et al* 1996; Bonsangue *et al* 1998), or a similarity on X (Heckmann 2007).

Next, a $\mathcal{Q}$-functor $f \colon X \to Y$ is any function that satisfies $f \cdot X \leqslant Y \cdot f$. This means that $X(x, y) \leqslant Y(fx, fy)$ for all $x, y \in X$.

$\mathcal{Q}$-categories and $\mathcal{Q}$-functors form a category $\mathcal{Q}\text{-}\mathbf{Cat}$. The full subcategory of $\mathcal{Q}\text{-}\mathbf{Cat}$ consisting of $\mathcal{Q}$-posets and $\mathcal{Q}$-functors, denoted $\mathcal{Q}\text{-}\mathbf{Pos}$, is isomorphic to the category $\mathbf{Pos}$ of posets if $\mathcal{Q} = \mathbf{2}$, and to the category $\mathbf{Met}$ of quasimetric spaces (resp. bounded by 1) if $\mathcal{Q} = [0, \infty]$ (resp. $\mathcal{Q} = [0, 1]$).

An important example of a $\mathcal{Q}$-poset is $\mathcal{Q}$ itself with the structure $\mathcal{Q}(a, b) := a \multimap b$.

Also, for any $\mathcal{Q}$-category X, its *dual* is defined as the set X together with the structure $X^{\circ} \colon X \longmapsto X$. It is denoted as X°.

$\mathcal{Q}\text{-}\mathbf{Cat}$ is a symmetric monoidal closed category, with tensor product:

$$X \otimes Y((x, y), (x', y')) = X(x, x') \otimes Y(y, y'),$$

and internal hom:

$$Y^X(f, g) = \bigwedge_{x \in X} Y(fx, gx).$$

In particular:

$$\widehat{X}(f,g) = \bigwedge_{x \in X} (fx \multimap gx), \tag{3}$$

where $\widehat{X}$ denotes $\mathcal{Q}^{X^\circ}$.

Recall that $g \colon X \to Y \dashv h \colon Y \to X$ in $\mathcal{Q}$-**Cat** iff $g \circ h \leq 1_Y$ and $h \circ g \geq 1_X$ in the pointwise order on Y^X iff $Y(gx,y) = X(x,hy)$ for all $x \in X$ and $y \in Y$.

2.4. $\mathcal{Q}$-*distributors*

In order theory one often encounters relations that "distribute over the order". To say it precisely, we say that a relation $\prec$ distributes over $\leq$ when $\prec$ equals both $\prec \cdot \leq$ and $\leq \cdot \prec$. An example of such a relation is the way-below relation $\ll$ known from domain theory. Now, if we lift the above definition to the $\mathcal{Q}$-setting, we have:

Definition 2.5. A $\mathcal{Q}$-distributor $\varphi \colon X {\longrightarrow\mkern-7mu\circ\,} Y$ is a $\mathcal{Q}$-relation $\varphi \colon X \longmapsto Y$ with

$$\varphi \cdot X = \varphi = Y \cdot \varphi.$$

For any fixed $\mathcal{Q}$, $\mathcal{Q}$-distributors form a category: The category $\mathcal{Q}$-**Dist** has $\mathcal{Q}$-categories as objects and $\mathcal{Q}$-distributors as morphisms. In particular, for any $\mathcal{Q}$-category X, its structure $X \colon X \longmapsto X$ is a $\mathcal{Q}$-distributor (written thereafter as $X \colon X {\longrightarrow\mkern-7mu\circ\,} X$) and assumes the role of the identity morphism on X in $\mathcal{Q}$-**Dist**. The composition of morphisms is inherited from $\mathcal{Q}$-**Rel**.

If $f \colon X \to Y$ is any function, then $f_* := Y \cdot f$ and $f^* := f^\circ \cdot Y$ are $\mathcal{Q}$-relations and one easily verifies that:

Fact 2.6. Let X, Y be $\mathcal{Q}$-categories. The following are equivalent:

(i) $f \colon X \to Y$ is a $\mathcal{Q}$-functor;
(ii) $f_* \colon X {\longrightarrow\mkern-7mu\circ\,} Y$ is a $\mathcal{Q}$-distributor;
(iii) $f^* \colon Y {\longrightarrow\mkern-7mu\circ\,} X$ is a $\mathcal{Q}$-distributor.

Observe that $X = 1_X^* = 1_{X*}$.

Furthermore, for any map $f \colon X \to Y$, in $\mathcal{Q}$-**Dist** we have the adjunction:

$$f_* \dashv f^*. \tag{4}$$

Note that $g \colon X {\longrightarrow\mkern-7mu\circ\,} Y \dashv h \colon Y {\longrightarrow\mkern-7mu\circ\,} X$ in $\mathcal{Q}$-**Dist** iff $g \cdot h \leqslant Y$ and $h \cdot g \geqslant X$.

It is well-known (see e.g. (Lawvere 2002), Sect.3) that the following are equivalent for any $\mathcal{Q}$-relation $\varphi \colon X \longmapsto Y$ between $\mathcal{Q}$-categories:

(i) $\varphi \colon X {\longrightarrow\mkern-7mu\circ\,} Y$ is a $\mathcal{Q}$-distributor;
(ii) $\varphi \colon X^\circ \otimes Y \to \mathcal{Q}$ is a $\mathcal{Q}$-functor.

We will most often use the above result in the following simplified form:

Fact 2.7. Let $\varphi\colon X \longrightarrow 1$ be a $\mathcal{Q}$-relation. The following are equivalent:

(i) $\varphi\colon X \longrightarrow 1$ is a $\mathcal{Q}$-distributor;
(ii) $\varphi\colon X° \to \mathcal{Q}$ is a $\mathcal{Q}$-functor.

(We shall issue a gentle warning here: from now on we sometimes treat distributors as $\mathcal{Q}$-functors and vice versa, without any further reminder.)

Let us immediately see an important example: since the identity $X\colon X \longrightarrow X$ is a $\mathcal{Q}$-distributor, it corresponds to the $\mathcal{Q}$-functor $X\colon X° \otimes X \to \mathcal{Q}$ whose exponential mate $\ulcorner X \urcorner\colon X \to \widehat{X}$ is the Yoneda $\mathcal{Q}$-functor $\mathsf{y}\colon X \to \widehat{X}$ defined as $\mathsf{y}x := X(-, x)$. (Recall that $\widehat{X}$ is defined in (3) and denotes the set of all $\mathcal{Q}$-functors of type $X° \to \mathcal{Q}$, or equivalently by Fact 2.7, the set of all $\mathcal{Q}$-distributors of type $X \longrightarrow 1$.)

One calls a $\mathcal{Q}$-functor $f\colon X \to Y$ fully faithful, if $X(x, z) = Y(fx, fz)$ for all $x, z \in X$; this condition is equivalent to $X = f^* \cdot f_*$, which is not hard to prove. The Yoneda embedding $\mathsf{y}\colon X \to \widehat{X}$ is fully faithful.

2.5. *Lower and upper closures for $\mathcal{Q}$-relations*

Definition 2.8. Let X, Y be $\mathcal{Q}$-posets. The *lower closure* of a $\mathcal{Q}$-relation $\phi\colon Y \longrightarrow X$ is $\downarrow\phi = \phi \cdot Y$, and the *upper closure* $\uparrow\phi = X \cdot \phi$

Thus for example:

$$\phi = \phi \cdot 1_Y \overset{(\mathbf{r})}{\leqslant} \phi \cdot Y = \downarrow\phi,$$

and

$$\downarrow\phi = \phi \cdot Y \overset{(\mathbf{r}),(\mathbf{t})}{=} \phi \cdot Y \cdot Y = \downarrow\downarrow\phi,$$

as expected from the lower closure.

Note that if $X = 1$, then (in a pointwise notation):

$$\downarrow\phi(z) = (\phi \cdot Y)(z) = \bigvee_{y \in Y} (Y(z, y) \otimes \phi y), \tag{5}$$

and so, if Y is a poset and $\phi \subseteq Y$, then (5) reduces to $z \in \downarrow\phi \iff \exists y \, (z \leqslant y \wedge y \in \phi)$, which is indeed the well-known definition of the lower closure.

Observe that a $\mathcal{Q}$-relation ϕ is a $\mathcal{Q}$-distributor iff $\phi = \downarrow\uparrow\phi$ (equivalently: $\phi = \uparrow\downarrow\phi$), hence the operation $\phi \mapsto \downarrow\uparrow\phi$ can be seen as a way of turning $\mathcal{Q}$-relations into $\mathcal{Q}$-distributors.

2.6. *The Yoneda Lemma*

Lemma 2.9 (Yoneda). Let X be a $\mathcal{Q}$-poset. For all $x \in X$ and $\psi \in \widehat{X}$, $\psi(x) = \widehat{X}(\mathsf{y}x, \psi)$.

Proof. Let $x \in X$.

$$
\begin{aligned}
\psi(x) &= X(x,x) \multimap \psi(x) && [\text{use } (\mathbf{r})] \\
&= \bigwedge_{y \in X}(X(y,x) \multimap \psi(y)) && [\text{put } y = x; \; \psi \colon X^{\circ} \to \mathcal{Q} \text{ is a } \mathcal{Q}\text{-functor}] \\
&= \widehat{X}(\mathsf{y}x, \psi) && [\text{def. } \widehat{X}]
\end{aligned}
$$

$\square$

Observe that the statement of the Yoneda Lemma for $\mathcal{Q} = \mathbf{2}$ reduces to:

$$
x \in \psi \iff {\downarrow}x \subseteq \psi,
$$

where in this case ψ is some subset of a poset X.

2.7. *Suprema*

Definition 2.10. Let X be a $\mathcal{Q}$-poset. A $\mathcal{Q}$-functor $\phi \in \widehat{X}$ has a supremum $\mathcal{S}\phi$ if for all $x \in X$,

$$
X(\mathcal{S}\phi, x) = \widehat{X}(\phi, \mathsf{y}x). \tag{6}
$$

In the poset case the above definition reduces to the well-known definition: a lower subset ϕ of a poset X has a supremum $\mathcal{S}\phi$ iff

$$
\forall x \in X \; (\mathcal{S}\phi \le x \iff \phi \subseteq {\downarrow}x).
$$

One easily proves a 'pointless' (as opposed to 'pointwise') version of (6): a $\mathcal{Q}$-functor $\phi \in \widehat{X}$ (considered as $\phi \colon 1 \relbar\joinrel\mapsto \widehat{X}$) has a supremum $x \in X$ iff

$$
\mathsf{y}^{*} \cdot \phi_{*} = x_{*}. \tag{7}
$$

2.8. *The generalized image-preimage adjunction*

Let $f \colon X \to Y$ be a $\mathcal{Q}$-functor and let $\underline{f}(\phi) = \phi \cdot f^{*}$ (the composition of $\mathcal{Q}$-distributors on the right), and $\widehat{f}(\psi) = \psi \cdot f_{*}$. Then by (4) we can calculate:

$$
(\underline{f} \circ \widehat{f})(\psi) = \psi \cdot f_{*} \cdot f^{*} \le \psi \cdot Y = \psi
$$

and

$$
(\widehat{f} \circ \underline{f})(\phi) = \phi \cdot f^{*} \cdot f_{*} \ge \phi \cdot X = \phi.
$$

We have proved that:

$$
\underline{f} \dashv \widehat{f} \tag{8}
$$

holds in $\mathcal{Q}$-**Cat** for the $\mathcal{Q}$-functors $\underline{f} \colon \widehat{X} \to \widehat{Y}$ and $\widehat{f} \colon \widehat{Y} \to \widehat{X}$.

Let us look at the above definitions pointwise:

$$
\underline{f}(\phi)(y) = \bigvee_{x \in X} (\phi x \otimes Y(y, fx)), \tag{9}
$$

$$
\widehat{f}(\psi)(x) = \psi(fx). \tag{10}
$$

One now can easily see that in the case of posets (9) and (10) reduce to, respectively, lower closure of the image of a monotone map f, and to the preimage of f. Consequently, for posets the adjunction (8) becomes:

$$\downarrow f[\phi] \subseteq \psi \iff \phi \subseteq f^{-1}[\psi]$$

for all lower subsets ϕ of X and all lower subsets ψ of Y. It is perhaps worth mentioning that the $\mathcal{Q}$-functor $\widehat{f}$ has a right adjoint as well as the left adjoint (see e.g. (Lawvere 2002), Sect.3. on Kan quantification) but we will not make any use of the former in this paper.

Finally note the following characterisation of $\mathcal{Q}$-functors from (Waszkiewicz 2009), which on the poset level corresponds to a well-known behavior of monotone functions with suprema of lower sets:

Proposition 2.11. Let X, Y be $\mathcal{Q}$-posets. For a function $f \colon X \to Y$ the following are equivalent:

(i) f is a $\mathcal{Q}$-functor;
(ii) $\underline{f} \circ \mathsf{y} \leqslant \mathsf{y} \circ f$;
(iii) $\mathcal{S}(\underline{f}(\phi)) \leqslant f(\mathcal{S}\phi)$ for all $\phi \in \widehat{X}$ such that the suprema on the left and right-hand side exist;
(iv) $\mathcal{S} \circ \underline{f} \circ \mathsf{y} = f$.

2.9. *Relative cocompleteness*

In our paper we use the notion of relative cocompleteness for $\mathcal{Q}$-categories developed in (Kelly 1982), (Albert and Kelly 1998), (Kock 1995), (Kelly and Schmitt 2005), (Kelly and Lack 2000), (Schmitt 2006) and explained in elementary terms in (Lai and Zhang 2007). Here is a summary of definitions needed for our paper.

Assume that a subcategory $\mathcal{I}$ of $\mathcal{Q}$-**Dist** is given and that it contains all $\mathcal{Q}$-distributors of the form f^* for a $\mathcal{Q}$-functor f, and satisfies

$$\text{if } y^* \cdot \varphi \in \mathcal{I} \text{ for all } y \in Y, \text{ then } \varphi \in \mathcal{I}. \tag{11}$$

This condition implies that $\mathcal{I}$ is fully determined by all $\mathcal{Q}$-distributors of type $X \nrightarrow 1$. Therefore, we will often only specify a set of $\mathcal{Q}$-distributors of type $X \nrightarrow 1$ for each $\mathcal{Q}$-category X, with the understanding that $\mathcal{I}$ is generated by this collection of $\mathcal{Q}$-distributors with help of (11). However, in this case we must check that the resulting $\mathcal{I}$ is indeed closed under composition.

We call a $\mathcal{Q}$-category X $\mathcal{I}$-cocomplete if X has all weighted colimits where the weight belongs to $\mathcal{I}$ (the definition of weighted colimits is standard, see e.g. (Lai and Zhang 2007)).

Note that in particular, by definitions above, the following hold:

(ax1) $x^* \in \mathcal{I}$ for all $x \colon 1 \to X$,

(ax2) $\phi \cdot f^* \in \mathcal{I}$ for all $\phi \in \mathcal{I}$ and all $\mathcal{Q}$-functors $f \colon X \to Y$,

which in pointwise notation is:

(ax1) $\mathsf{y}x \in \mathcal{I}$ for all $x \colon 1 \to X$;
(ax2) $\underline{f}(\phi) \in \mathcal{I}$ for all $\phi \in \mathcal{I}$ and all $\mathcal{Q}$-functors $f \colon X \to Y$.

For a $\mathcal{Q}$-poset X, we put

$$\mathcal{I}(X) = \{\psi \colon X \longrightarrow 1 \mid \psi \in \mathcal{I}\}$$

and

$$\mathcal{I}_S(X) = \{\psi \in \mathcal{I}(X) \mid \psi \text{ has a supremum in } X\},$$

both considered as $\mathcal{Q}$-subcategories of $\widehat{X}$. For any $x \in X$, $x^* \in \mathcal{I}(X)$, hence the Yoneda functor $\mathsf{y}x = x^*$ is of type $X \to \mathcal{I}(X)$. Since $\mathsf{y}^* \cdot (\mathsf{y}x)_* = \mathsf{y}^* \cdot \mathsf{y}_* \cdot x_* = x_*$ for any $x \in X$, then by (7) we can further corestrict the Yoneda functor and obtain a $\mathcal{Q}$-functor $\mathsf{y} \colon X \to \mathcal{I}_S(X)$. On the other hand, there is a map $\mathcal{S} \colon \mathcal{I}_S(X) \to X$ which to each $\psi \in \mathcal{I}_S(X)$ assigns its supremum. It is in fact a $\mathcal{Q}$-functor. Clearly then, $\mathsf{y}^* \cdot \psi_* = (\mathcal{S}(\psi))_* = \mathcal{S}_* \cdot \psi_*$ for any $\psi \in \mathcal{I}_S(X)$, i.e. $\mathsf{y}^* = \mathcal{S}_*$, which is equivalent to say that $\mathcal{S}$ is a left adjoint to the restriction $\mathsf{y} \colon X \to \mathcal{I}_S(X)$ of the Yoneda $\mathcal{Q}$-functor: $\mathcal{S} \dashv \mathsf{y}$. Using these observations we can easily deduce an alternative definition of $\mathcal{I}$-cocompleteness:

Definition 2.12. A $\mathcal{Q}$-poset X is $\mathcal{I}$-*cocomplete* iff there exists a $\mathcal{Q}$-functor $\mathcal{S} \colon \mathcal{I}X \to X$ such that

$$X(\mathcal{S}\phi, x) = \widehat{X}(\phi, \mathsf{y}x)$$

for any $\phi \in \mathcal{I}X$ and $x \in X$.

Definition 2.13. A $\mathcal{Q}$-distributor ϕ is a $\mathcal{Q}$-ideal on X if $\phi \in \mathcal{I}X$.

Example 2.14. For any $\mathcal{Q}$, we can choose $\mathcal{I} = \mathcal{Q}$-**Dist**. Then $\mathcal{I}$-cocomplete means cocomplete.

Example 2.15. For $\mathcal{Q} = \mathbf{2}$, we consider all $\mathcal{Q}$-distributors of type $X \longrightarrow 1$ corresponding to order-ideals in X, which indeed generate a subcategory $\mathcal{I}$ of $\mathcal{Q}$-**Dist**. Then an ordered set X is $\mathcal{I}$-cocomplete if and only if each order-ideal on X has a supremum.

Example 2.16. For $\mathcal{Q} = [0, \infty]$ or $\mathcal{Q} = [0, 1]$, we consider all $\mathcal{Q}$-distributors of type $X \longrightarrow 1$ corresponding to ideals in X in the sense of (Bonsangue *et al* 1998). Again, they generate a subcategory of $\mathcal{Q}$-**Dist**. These ideals correspond to equivalence classes of forward Cauchy sequences on X. Hence, X is $\mathcal{I}$-cocomplete if and only if each forward Cauchy sequence on X converges.

Example 2.17. For any $\mathcal{Q}$ we can choose $\mathcal{I}$ to be the subcategory of $\mathcal{Q}$-**Dist** consisting of all right adjoint $\mathcal{Q}$-distributors. Recall from (Lawvere 2002) that, for $\mathcal{Q} = [0, \infty]$ and for $\mathcal{Q} = [0, 1]$, a right adjoint $\mathcal{Q}$-distributor $X \longrightarrow 1$ corresponds to an equivalence class of Cauchy sequences on X. A metric space X is $\mathcal{I}$-cocomplete if each Cauchy sequence on X converges.

Example 2.18. For any completely distributive quantale $\mathcal{Q}$ and any $\mathcal{Q}$-category X, a $\mathcal{Q}$-distributor $\psi \colon X \longrightarrow 1$ is a FSW-ideal if:

1. $\bigvee_{z \in X} \psi(z) = \mathbf{1}$, and
2. for all $\varepsilon_1, \varepsilon_2, \delta \prec \mathbf{1}$, for all $x_1, x_2 \in X$, whenever $\varepsilon_1 \prec \psi(x_1)$ and $\varepsilon_2 \prec \psi(x_2)$, then there exists $z \in X$ such that $\delta \prec \psi(z)$, $\varepsilon_1 \prec \psi(x_1) \otimes X(x_1, z)$ and $\varepsilon_2 \prec \psi(x_2) \otimes X(x_2, z)$.

One easily checks that the generated class is indeed a subcategory of $\mathcal{Q}$-**Dist**. Note that for $\mathcal{Q} = [0, \infty]$ and for $\mathcal{Q} = [0, 1]$ FSW-ideals on X are in a bijective correspondence with equivalence classes of forward Cauchy nets on X (Flagg *et al* 1996); for $\mathcal{Q} = \mathbf{2}$, FSW-ideals are characteristic maps of order-ideals on X. Therefore this example unifies Examples 2.15, 2.16.

Example 2.19. For any $\mathcal{Q}$ we can choose $\mathcal{I}$ to be the subcategory of $\mathcal{Q}$-**Dist** consisting of all $\mathcal{Q}$-distributors of type $X \xrightarrow{} 1$ that satisfy:

1. $\bigvee_x \phi(x) = \mathbf{1}$, and
2. $\phi(x) \otimes \phi(y) \leqslant \bigvee_z (\phi(z) \otimes X(x, z) \otimes X(y, z))$.

This definition appears in (Zhang and Fan 2005) and (Lai and Zhang 2007).

Further examples are mentioned in (Schmitt 2006; Vickers 2005).

It is important to notice that $\mathcal{Q}$ is itself an $\mathcal{I}$-cocomplete $\mathcal{Q}$-poset. Indeed, if $\psi \in \mathcal{I}\mathcal{Q}$ and $a \in Q$, then the calculation:

$$
\begin{aligned}
\mathcal{Q}(\textstyle\bigvee_b (\phi b \otimes b), a) &= \textstyle\bigwedge_b (\phi b \multimap (b \multimap a)) \\
&= \textstyle\bigwedge_b (\phi b \multimap \mathsf{y}(a)(b))) \\
&= \widehat{\mathcal{Q}}(\phi, \mathsf{y}a)
\end{aligned}
$$

shows that we can define $\mathcal{S}\phi := \bigvee_b (\phi b \otimes b)$. One can also show that since unit is the top element of Q, then $\mathcal{S}\phi = \phi(\mathbf{1})$, which gives an alternative description of supremum.

2.10. *Scott-continuous $\mathcal{Q}$-functors*

Definition 2.20. A $\mathcal{Q}$-functor $f : X \to Y$ is *Scott-continuous* if, for all $\phi \in \mathcal{I}X$, where ϕ has a supremum in X, $\underline{f}(\phi)$ has a supremum in Y, and, moreover,

$$f(\mathcal{S}\phi) = \mathcal{S}(\underline{f}(\phi)). \tag{12}$$

Equivalently, f is Scott-continuous if and only if f preserves all colimits with weight in $\mathcal{I}$ which exist in X. Clearly, every cocontinuous $\mathcal{Q}$-functor is Scott-continuous. For $\mathcal{Q} = \mathbf{2}$ and FSW-ideals Scott-continuity has the usual meaning. For $\mathcal{Q} = [0, 1]$ or $\mathcal{Q} = [0, \infty]$ and the choice of $\mathcal{Q}$-ideals from Example 2.16, f is Scott-continuous if and only if f preserves limits of forward Cauchy sequences.

Let us denote by $\mathcal{Q}$-**Cocont** the category of $\mathcal{I}$-cocomplete $\mathcal{Q}$-posets with Scott-continuous $\mathcal{Q}$-functors as morphisms. Then the following are well-known: see e.g. (Albert and Kelly 1998), (Kelly 1982), (Kelly and Schmitt 2005), also Thm. 4.7. of (Lai and Zhang 2007):

Proposition 2.21. Let Y be a $\mathcal{I}$-cocomplete category. Then any $\mathcal{Q}$-functor $f : X \to Y$ uniquely extends to a Scott-continuous $\mathcal{Q}$-functor $F : \mathcal{I}(X) \to Y$.

Theorem 2.22. The inclusion functor $\mathcal{Q}$-**Cocont** $\to \mathcal{Q}$-**Cat** has a left adjoint which sends a $\mathcal{Q}$-category X to $\mathcal{I}(X)$ and a $\mathcal{Q}$-functor f to $\underline{f}$.

3. The limit-colimit coincidence for $\mathcal{Q}$-Cocont

Our goal in this paper is to show a generalisation of Theorem 3.3.7 of (Abramsky and Jung 1994) described there as: *one of the fundamental tools of domain theory, which plays its most prominent role in the solution of recursive domain equations.* Our proof is slightly subtler that the original one but otherwise follows it quite closely. Let us repeat that the generalization is made on two levels: we replace posets with $\mathcal{Q}$-posets, and furthermore, our version of the result is relative to the definition of ideals.

Let us recall that every $\mathcal{Q}$-poset X is also a partial order since we can define:

$$x \leqslant_X y \ \text{ iff } \ X(x, y) = \mathbf{1}.$$

The statement and the proof of our theorem crucially use the induced order. This has the consequence that our theorem will trivialize whenever the order trivializes, for example when we work with $[0, 1]$-**Cocont** and choose our diagram to consist of complete metric spaces. However, for nonsymmetric spaces the theorem shows its full strength.

Finally, in order for the proof to work we must have at our disposal an ideal over the set indexing the diagram. The easiest solution that would guarantee existence of such an ideal is the condition that explicitly augments (ax1)-(ax2):

> The $\mathcal{Q}-$functor $\psi \colon \mathbb{N}^\circ \to \mathcal{Q}$ given by $\psi(n) := \mathbf{1}$ for all $n \in \mathbb{N}$ is an ideal. (ax3)

Here $\mathbb{N}$ is the $\mathcal{Q}$-poset defined as $\mathbb{N}(n, m) := \mathbf{1}$ iff $n \leq m$, and $\mathbb{N}(n, m) := \bot$ otherwise. One can show that (ax3) is satisfied by any of the concrete definitions of ideals that we discussed in Section 2.9. We assume (ax3) from now on.

Lemma 3.1. Let X, Y be $\mathcal{Q}$-posets and assume that Y is $\mathcal{I}$-cocomplete. Suppose that $(f_n \colon X \to Y)_{n \in \mathbb{N}}$ is a chain of $\mathcal{Q}$-functors, i.e. for all $n \in \mathbb{N}$, $f_n \leqslant f_{n+1}$ with respect to the induced order $\leqslant$ on Y^X. Then:

(i) $f := \bigvee_{n \in \mathbb{N}} f_n$ exists and is given by $f(x) := \mathcal{S}(\underline{F_x}(\psi))$, for $x \in X$, where ψ is chosen as in (ax3), and $F_x \colon \mathbb{N} \to Y$ is $F_x(n) := f_n(x)$;

(ii) furthermore, $Y(fx, y) = \bigwedge_{n \in \mathbb{N}} Y(f_n(x), y)$ for all $x \in X$ and $y \in Y$;

(iii) furthermore, if all f_n's are Scott-continuous, then f is Scott-continuous, too.

Proof. Fix $x \in X$. Since $F_x(n) = f_n(x) \leqslant f_m(x) = F_x(m)$ for all $n \leqslant m$, then $F_x \colon \mathbb{N} \to Y$ is a $\mathcal{Q}$-functor. Therefore by (ax2), $\underline{F_x}(\psi)$ is an ideal on Y, and since Y is $\mathcal{I}$-cocomplete, the supremum $\mathcal{S}(\underline{F_x}(\psi))$ exists and is an element of Y. Now, if $x, z \in X$, then $Y(fx, fz) \geqslant \widehat{Y}(\underline{F_x}(\psi), \underline{F_z}(\psi)) = \bigwedge_{n \in \mathbb{N}} \underline{F_z}(\psi)(F_x(n)) = \bigwedge_{n \in \mathbb{N}} \bigvee_{m \in \mathbb{N}} Y(F_x(n), F_z(m)) \geqslant \bigwedge_{n \in \mathbb{N}} Y(F_x(n), F_z(n)) = \bigwedge_{n \in \mathbb{N}} Y(f_n(x), f_n(z)) \geqslant X(x, z)$. Therefore f is a $\mathcal{Q}$-functor. For (ii), $Y(fx, y) = \widehat{\mathbb{N}}(\psi, \widehat{F_x}(yy)) = \bigwedge_n \widehat{F_x}(yy)(n) = \bigwedge_n Y(f_n(x), y)$ for all $x \in X$ and $y \in Y$.

Clearly for $n \in \mathbb{N}$, $Y^X(f_n, f) = \bigwedge_{x \in X} Y(f_n(x), fx) \geqslant \bigwedge_{x \in X} \bigwedge_{n \in \mathbb{N}} Y(f_n(x), fx) = \bigwedge_{x \in X} Y(fx, fx) = \mathbf{1}$, hence $f_n \leqslant f$ for all $n \in \mathbb{N}$. Now, suppose that $f_n \leqslant g$, $n \in \mathbb{N}$, for some $g \colon X \to Y$. Then $\mathbf{1} = \bigwedge_{x \in X} \bigwedge_{n \in \mathbb{N}} Y(f_n(x), g(x)) = \bigwedge_{x \in X} Y(fx, gx) = Y^X(f, g)$. This proves that $f = \bigvee_{n \in \mathbb{N}} f_n$, i.e. part (i).

Lastly, suppose that all f_n's are Scott-continuous. Fix $y \in Y$ and $\phi \in \mathcal{I}(X)$ that has a supremum $\mathcal{S}\phi$ in X. Then $Y(f(\mathcal{S}\phi), y) = \bigwedge_n Y(f_n(\mathcal{S}\phi), y) = \bigwedge_n Y(\mathcal{S}\underline{f_n}(\phi)), y) = \bigwedge_{x \in X}(\phi x \multimap \bigwedge_n Y(f_n(x), y)) = \widehat{X}(\phi, \widehat{f}(\mathsf{y}y)) = Y(\mathcal{S}\underline{f}(\phi), y)$. Since y is arbitrary, $f(\mathcal{S}\phi) = \mathcal{S}\underline{f}(\phi)$, as required. $\qquad\square$

Theorem 3.2. Consider the expanding sequence in $\mathcal{Q}$-**Cocont**:

$$D_0 \underset{p_{01}}{\overset{e_{10}}{\rightleftarrows}} D_1 \underset{p_{12}}{\overset{e_{21}}{\rightleftarrows}} D_2 \underset{p_{23}}{\overset{e_{32}}{\rightleftarrows}} D_3 \rightleftarrows \cdots \qquad\qquad\text{(a)}$$

where (b)-(f) hold for all $m \leq n \leq k \in \mathbb{N}$:

(b) $p_{nn} = e_{nn} = 1_{D_n}$,
(c) $e_{kn} \circ e_{nm} = e_{km}$,
(d) $p_{mn} \circ p_{nk} = p_{mk}$,
(e) $e_{nm} \circ p_{mn} \leqslant 1_{D_n}$ (for convenience by $\leqslant$ we denote the induced order $\leqslant_{[D_n \to D_n]}$),
(f) $p_{mn} \circ e_{nm} = 1_{D_m}$.

Then there exists an $\mathcal{I}$-cocomplete $\mathcal{Q}$-poset D together with morphisms:

$$D_n \underset{p_n}{\overset{e_n}{\rightleftarrows}} D$$

for all $n \in \mathbb{N}$ such that:

(g) D is the limit of the diagram of projections:

$$D_0 \xleftarrow{p_{01}} D_1 \xleftarrow{p_{12}} D_2 \xleftarrow{p_{23}} D_3 \xleftarrow{\quad} \cdots , \qquad\qquad\text{(†)}$$

(h) D is the colimit of the diagram of embeddings:

$$D_0 \xrightarrow{e_{10}} D_1 \xrightarrow{e_{21}} D_2 \xrightarrow{e_{32}} D_3 \xrightarrow{\quad} \cdots , \qquad\qquad\text{(††)}$$

(i) for all $n \in \mathbb{N}$, $e_n \circ p_n \leqslant 1_D$,
(j) for all $n \in \mathbb{N}$, $p_n \circ e_n = 1_{D_n}$.

Remark: Note that assumptions (e) and (f) state (in the nomenclature of domain theory) that for every $n, m \in \mathbb{N}$, maps

$$e_{nm} \colon D_m \rightleftarrows D_n \colon p_{mn}$$

form e-p pairs; in particular $e_{nm} \dashv p_{mn}$.

Proof. We start by a general observation that the inclusion functor $\mathcal{Q}\text{-}\mathbf{Cocont} \to \mathcal{Q}\text{-}\mathbf{Cat}$ not only has a left adjoint (Thm. 2.22) but is monadic. Since $\mathcal{Q}\text{-}\mathbf{Cat}$ is complete, $\mathcal{Q}\text{-}\mathbf{Cocont}$ is complete as well. Thus we know that the limit of (†) exists, and in fact is given by:

$$D := \left\{ (x_0, x_1, ...) \in \prod_{i \in \mathbb{N}} D_i \ \middle| \ p_{ii+1}(x_{i+1}) = x_i \right\}.$$

For completeness of the argument, we include an explicit proof of this fact:

Clearly, D is a $\mathcal{Q}$-poset when considered with the structure inherited from the product $\prod_i D_i$. Next, D is $\mathcal{I}$-cocomplete. Indeed, take $\phi \in \mathcal{I}D$. We claim that $\mathcal{S}\underline{\iota}(\phi)$ is the supremum of ϕ. Here $\iota \colon D \to \prod_i D_i$ is the inclusion map (clearly ι is a fully faithful $\mathcal{Q}$-functor). By (ax2), $\underline{\iota}(\phi) \in \mathcal{I}(\prod_i D_i)$. Since $\prod_i D_i$ is $\mathcal{I}$-cocomplete, $\mathcal{S}\underline{\iota}(\phi)$ exists. Then:

$$
\begin{aligned}
p_{ii+1}(\pi_{i+1}(\mathcal{S}\underline{\iota}(\phi))) &= \mathcal{S}(\underline{p_{ii+1}}(\underline{\pi_{i+1}}(\underline{\iota}(\phi)))) &&\text{[projections are Scott-continuous]} \\
&= \mathcal{S}(\underline{p_{ii+1} \circ \pi_{i+1} \circ \iota}(\phi)) &&\text{[$\underline{(\cdot)}$ is a functor]} \\
&= \mathcal{S}(\underline{\pi_i \circ \iota}(\phi)) &&\text{[def. of D]} \\
&= \pi_i(\mathcal{S}\underline{\iota}(\phi)) &&\text{[π_i is Scott-continuous]}.
\end{aligned}
$$

Therefore $\mathcal{S}\underline{\iota}(\phi) \in D$. Moreover, for any $x \in D$:

$$
\begin{aligned}
D(\mathcal{S}\underline{\iota}(\phi), x) &= \textstyle\prod_i D_i(\mathcal{S}\underline{\iota}(\phi), \iota x) &&\text{[def. of D]} \\
&= \widehat{\textstyle\prod_i D_i}(\underline{\iota}(\phi), \mathsf{y}(\iota x)) &&\text{[$\prod_i D_i$ is $\mathcal{I}$-cocomplete]} \\
&= \widehat{\textstyle\prod_i D_i}(\underline{\iota}(\phi), \underline{\iota}(\mathsf{y}x)) &&\text{[$\underline{\iota} \circ \mathsf{y} = \mathsf{y} \circ \iota$]} \\
&= \widehat{D}(\phi, \mathsf{y}x) &&\text{[$\underline{\iota}$ is fully faithful]}
\end{aligned}
$$

We can now conclude that $\mathcal{S}\underline{\iota}(\phi)$ is the supremum of ϕ and hence that D is an $\mathcal{I}$-cocomplete $\mathcal{Q}$-poset.

Secondly, for every $i \in \mathbb{N}$ we define $p_i \colon D \to D_i$ by:

$$p_i := \pi_i \mid_D . \tag{13}$$

Observe that $(D, p_i \colon D \to D_i)$ is a cone over the diagram (†), since p_i's are Scott-continuous $\mathcal{Q}$-functors and they commute with p_{mn} $(m \leq n)$ by the definition of D. Suppose now that $(Z, g_i \colon Z \to D_i)$ is another cone. Then it is easy to see that the unique mediating morphism $g \colon Z \to D$ is given by

$$g(z) := (g_0(z), g_1(z), ...)$$

The map g is a composition of Scott-continuous $\mathcal{Q}$-functors, so it is itself a Scott-continuous $\mathcal{Q}$-functor. We have now proved (g), i.e. that D is the limit of the diagram (†) of projections.

We now claim that for

$$\tau_{ji} := \begin{cases} p_{ji} & \text{for } j \leq i, \\ e_{ji} & \text{for } j > i \end{cases}$$

$$(D_i, \tau_{ji} \colon D_i \to D_j) \quad \text{is a cone over } (\dagger) \text{ for all } i \in \mathbb{N}. \tag{14}$$

In other words, we need to show commutativity of all triangles of the form:

$$
\begin{array}{c}
D_i \\
\tau_{ki} \Big\downarrow \quad \searrow \tau_{li} \\
D_0 \longleftarrow \cdots \longleftarrow D_k \xleftarrow{\; p_{kl} \;} D_l \longleftarrow \cdots
\end{array}
$$

Fix $k < l$. If $l \leq i$, then $p_{kl} \circ \tau_{li} = p_{kl} \circ p_{li} = p_{ki} = \tau_{ki}$. If $k \leq i < l$, then $p_{kl} \circ \tau_{li} = p_{kl} \circ e_{li} = p_{ki} \circ (p_{il} \circ e_{li}) = p_{ki} = \tau_{ki}$. Otherwise $i < k$ and then $p_{kl} \circ \tau_{li} = (p_{kl} \circ e_{lk}) \circ e_{ki} = e_{ki} = \tau_{ki}$. This proves (14).

Since each D_i is a cone and D is a limit of $(\dagger)$, then for all $i \in \mathbb{N}$ there exists a unique morphism $e_i \colon D_i \to D$ such that

$$
\begin{array}{ccc}
D_i & \xrightarrow{\;\tau_{ji}\;} & D_j \\
e_i \,\Big\vdots & \nearrow p_i & \\
D & &
\end{array}
$$

commutes, i.e. for all $i \in \mathbb{N}$:

$$e_i(x) := (\tau_{ji}(x))_{j \in \mathbb{N}}. \tag{15}$$

Our next goal is to demonstrate that $(D, e_i \colon D_i \to D)$ is a cocone over the diagram of embeddings $(\dagger\dagger)$, i.e. that every diagram for $k < l$ (the case $k = l$ is clear) commutes:

$$
\begin{array}{c}
D \\
e_k \Big\uparrow \quad \nwarrow e_l \\
D_k \xrightarrow[\; e_{lk} \;]{} D_l
\end{array}
$$

or that for all $j \in \mathbb{N}$:

$$\tau_{jl} \circ e_{lk} = \tau_{jk}. \tag{16}$$

For $l < j$, $\tau_{jl} \circ e_{lk} = e_{jl} \circ e_{lk} = e_{jk} = \tau_{jk}$. For $k < j \leq l$, $\tau_{jl} \circ e_{lk} = p_{jl} \circ e_{lk} = (p_{jl} \circ e_{lj}) \circ e_{jk} = e_{jk} = \tau_{jk}$. For $j \leq k$, $\tau_{jl} \circ e_{lk} = p_{jl} \circ e_{lk} = p_{jk} \circ (p_{kl} \circ e_{lk}) = p_{jk} = \tau_{jk}$. Therefore (16) holds.

Next step is to show (i): Let $z = (z_j)_{j \in \mathbb{N}} \in D$. Then $(e_n \circ p_n)(z) = e_n(z_n) = (\tau_{jn}(z_n))_{j \in \mathbb{N}}$. However for $j \leq n$, $\tau_{jn}(z_n) = p_{jn}(z_n) = z_j$ and for $j > n$, $\tau_{jn}(z_n) = e_{jn}(z_n) = e_{jn}(p_{nj}(z_j)) \overset{(e)}{\leqslant} z_j$. In any case, we can conclude $(e_n \circ p_n)(z) \leqslant (z_j)_{j \in \mathbb{N}} = z$, as required.

Part (j) is clear, since $p_n \circ e_n = \pi_n \mid_D ((\tau_{jn})_{j \in \mathbb{N}}) = \tau_{nn} = p_{nn} \overset{(b)}{=} 1_{D_n}$.

It now remains to see (h), i.e. that D is the colimit of the diagram $(\dagger\dagger)$ of embeddings.

We define a family of functions indexed by elements of D. Let $x \in D$. Then $F_x \colon \mathbb{N} \to D$ is given by:

$$F_x(n) := (e_n \circ p_n)(x).$$

Since for $n \leq m$, $e_n \circ p_n = e_m \circ e_{mn} \circ p_{nm} \circ p_m \leqslant e_m \circ p_m$, the collection $(e_n \circ p_n)_{n \in \mathbb{N}}$ is a chain in D^D. Therefore, by the proof of Lemma 3.1, F_x is a $\mathcal{Q}$-functor.

Next, we will prove that (for ψ as in (ax3)):

$$\mathcal{S}(\underline{F_x}(\psi)) = x. \tag{17}$$

For,

$$
\begin{aligned}
p_n(\mathcal{S}(\underline{F_x}(\psi))) &= \mathcal{S}(\underline{p_n \circ F_x}(\psi)) & [p_n \text{ is Scott-continuous}] \\
&= \mathcal{S}(\textstyle\bigvee_i (\psi(i) \otimes D_n(-, p_n \circ F_x(i)))) & [\text{def. } \underline{p_n \circ F_x}] \\
&= \mathcal{S}(\textstyle\bigvee_i (D_n(-, p_n \circ e_i \circ p_i(x)))) & [\text{def. } \underline{F_x}, \text{ def. } \psi] \\
&= \mathcal{S}(D_n(-, p_n(x))) & [p_n \circ e_i \circ p_i = p_n \text{ for all } n \leqslant i \in \mathbb{N}] \\
&= \mathcal{S}(\mathsf{y}(p_n(x))) & [\text{def. } \mathsf{y}] \\
&= p_n(x) & [\mathcal{S} \dashv \mathsf{y}]
\end{aligned}
$$

for all $n \in \mathbb{N}$, and hence (17) holds.

Take any other cocone $(g_i \colon D_i \to G \mid i \in \mathbb{N})$ of the diagram (††) of embeddings. We thus have for all $i \leqslant j$:

$$g_j \circ e_{ji} = g_i. \tag{18}$$

We shall prove that the unique mediating morphism $g \colon D \to G$ is given by:

$$g(x) := \mathcal{S}(\underline{H_x}(\psi))$$

where $H_x \colon \mathbb{N} \to G$ is

$$H_x(n) := g_n \circ p_n(x).$$

By (18) it is easy to see that $(g_n \circ p_n)_{n \in \mathbb{N}}$ is a chain in G^D. Therefore, by Lemma 3.1, H_x is a $\mathcal{Q}$-functor, $\underline{H_x}(\psi)$ is an ideal, and $g \colon D \to G$ is a $\mathcal{Q}$-functor given by $g = \bigvee_n (g_n \circ p_n)$. Moreover, since all maps $g_n \circ p_n$, $n \in \mathbb{N}$, are Scott-continuous, g is Scott-continuous, too.

We need to show that for all $i \in \mathbb{N}$, $g_i = g \circ e_i$. Let $x \in D_i$.

$$
\begin{aligned}
g \circ e_i(x) &= \mathcal{S}(\underline{H_{e_i(x)}}(\psi)) & [\text{def. } g] \\
&= \mathcal{S}(\textstyle\bigvee_{n \in \mathbb{N}} (\psi(n) \otimes G(-, H_{e_i(x)}(n)))) & [\text{def. } \underline{H_{e_i(x)}}] \\
&= \mathcal{S}(\textstyle\bigvee_{n \in \mathbb{N}} G(-, g_n \circ p_n \circ e_i(x))) & [\text{def. } \psi, \text{ def. } H_{e_i(x)}] \\
&= \mathcal{S}(G(-, g_i(x))) & [g_n \circ p_n \circ e_i \leqslant g_i, \text{ also } n = i] \\
&= \mathcal{S}\mathsf{y}(g_i(x)) & [\text{def. } \mathsf{y}] \\
&= g_i(x). & [\mathcal{S} \circ \mathsf{y} = 1]
\end{aligned}
$$

We will finish the proof by showing that the map $g \colon D \to G$ is unique. Suppose that $g' \colon D \to G$ satisfies $g' \circ e_i = g_i$ for all $i \in \mathbb{N}$. Then $g' \circ e_i \circ p_i(x) = g_i \circ p_i(x)$ for all

$x \in D_i$. By definitions of F_x and H_x, this is the same as: $g' \circ F_x(i) = H_x(i)$, and consequently, $\mathcal{S}\underline{g' \circ F_x}(\psi) = \mathcal{S}\underline{H_x}(\psi)$. By Scott-continuity and (17), $g'(x) = g'(\mathcal{S}\underline{F_x}(\psi)) = \mathcal{S}\underline{g' \circ F_x}(\psi) = \mathcal{S}\underline{H_x}(\psi) = g(x)$. □

For FSW-ideals (Example 2.18) and $\mathcal{Q} = [0, \infty]$ we obtain the following:

Corollary 3.3. The category of forward Cauchy complete quasi-metric spaces and non-expansive maps has bilimits of expanding sequences.

4. Future work

There is some evidence that parts of domain theory can also be lifted to yet more general setup of $(\mathbb{T}, \mathcal{Q})$-categories introduced by (Clementino and Hofmann 2003; Clementino and Tholen 2003), who built on earlier work of (Barr 1970) and (Manes 1974). In $(\mathbb{T}, \mathcal{Q})$-categories the quantale $\mathcal{Q}$ is made compatible with a **Set**-monad $\mathbb{T}$. This type of further generalization of $\mathcal{Q}$-categories offers considerable research challenges but, on the other hand, promises some satisfactory general results. For example the category of $(\mathbb{U}, \mathbf{2})$-categories ($\mathbb{U}$ is the ultrafilter monad) is isomorphic to **Top**, the category of topological spaces. It seems plausible that a systematic research in this direction will result in a deeper understanding of the links between topology, order and distance.

References

Abramsky, S. and Jung, A. (1994) Domain Theory. In S.Abramsky, D.M. Gabbay and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science* **3**, pp. 1–168, Oxford University Press.

Albert, M.H. and Kelly, G.M. (1998) The closure of a class of colimits, *Journal of Pure and Applied Algebra* **51**, pp. 1-17.

America, P. and Rutten, J.J.M.M (1989) Solving Reflexive Domain Equations in a Category of Complete Metric Spaces, *Journal of Computer and System Sciences* **39**(3), pp. 343–375.

Barr, M. (1970) Relational Algebras, in Reports of the Midwest Category Seminar, IV, *Lecture Notes in Mathematics* **137**, Springer, pp. 39–55.

Bonsangue, M.M., van Breugel, F. and Rutten, J.J.M.M. (1998) Generalized Metric Spaces: Completion, Topology, and Powerdomains via the Yoneda Embedding, *Theoretical Computer Science* **193**(1-2), pp. 1–51.

Clementino, M.M. and Hofmann, D. (2003) Topological Features of Lax Algebras, *Applied Categorical Structures* **11**, pp. 267–286.

Clementino, M.M. and Tholen, W. (2003) Metric, Topology and Multicategory — a Common Approach, *Journal of Pure and Applied Algebra* **179**, pp. 13–47.

Clementino, M.M., Hofmann, D. and Tholen, W. (2004) One Setting for All: Metric, Topology, Uniformity, Approach Structure, *Applied Categorical Structures* **12** no. 2, pp. 127-154.

Flagg, R.C. (1997) Quantales and Continuity Spaces, *Algebra Universalis* **37**, pp. 257–276.

Flagg, R.C., Kopperman, R. (1995) Fixed Points and Reflexive Domain Equations in Categories of Continuity Spaces. *Electronic Notes in Theoretical Computer Science* **1**.

Flagg, R.C., Kopperman, R. (1997) Continuity Spaces: Reconciling Domains and Metric Spaces, *Theoretical Computer Science* **177**(1), pp. 111-138.

Flagg, R.C., Sünderhauf, P. (2002) The Essence of Ideal Completion in Quantitative Form. *Theoretical Computer Science* **278**(1-2), pp. 141–158.

Flagg, R.C., Sünderhauf, P. and Wagner, K.R. (1996) A Logical Approach to Quantitative Domain Theory, *Topology Atlas Preprint no. 23*, available on-line at: http://at.yorku.ca/e/a/p/p/23.htm

Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M. and Scott, D.S. (2003) Continuous Lattices and Domains. Volume 93 of *Encyclopedia of mathematics and its applications*, Cambridge University Press.

Heckmann, R. (2007) Similarity, Topology, and Uniformity, presented at Dagstuhl workshop "Logic, Computability and Topology in Computer Science: A New Perspective for Old Disciplines", submitted. Available on-line at: http://rw4.cs.uni-sb.de/ heckmann/domains.html

Hofmann, D. (2007) Topological Theories and Closed Objects, *Advances in Mathematics* **215**, pp. 789-824.

Kelly, G.M. (1982) Basic Concepts of Enriched Category Theory, in: London Mathematical Society Lecture Notes **64**, Cambridge Univ. Press.

Kelly, G.M. and Lack, S. (2000) On the Monadicity of Categories with Chosen Colimits, *Theory and Applications of Categories* **7**, pp. 148-170.

Kelly, G.M. and Schmitt, V. (2005) Notes on Enriched Categories with Colimits of Some Class, *Theory and Applications of Categories* **14**, pp. 399-423.

Kock, A. (1995) Monads for which Structures are Adjoint to Units, *Journal of Pure and Applied Algebra* **104**, pp. 41-59.

Lai, H. and Zhang, D. (2007) Complete and Directed-complete Ω-categories, *Theoretical Computer Science* **388**, pp. 1-25.

Lawvere, F.W. (2002) Metric Spaces, Generalized Logic and Closed Categories, *Reprints in Theory and Applications of Categories* **1**, pp. 1–37.

Manes, E.G. (1974) Compact Hausdorff Objects, *General Topology and Applications* **4**, pp. 341-360.

Rosenthal, K. (1990) Quantales and Their Applications, Longman, Harlow, Pittman Research Notes in Mathematics series vol. **234**.

Rutten, J.J.M.M. (1996) Elements of Generalized Ultrametric Domain Theory, *Theoretical Computer Science* **170**, pp. 349–381.

Schmitt, V. (2006) Flatness, Preorders and General Metric Spaces, *Georgian Mathematical Journal*, to appear, arXiv:math.CT/0602463.

Vickers, S. (2005) Localic Completion of Generalized Metric Spaces I, *Theory and Applications of Categories* **14**, pp. 328–356.

Wagner, K.R. (1994) Solving Recursive Domain Equations with Enriched Categories, PhD Thesis, Carnegie Mellon University.

Waszkiewicz, P. (2009) On Domain Theory over Girard Quantales, *Fundamenta Informaticae* **92**, pp. 1–24.

Zhang, Q.-Y., Fan, L. (2005) Continuity in Quantitative Domains, *Fuzzy Sets and Systems* **154**(1), pp. 118–131.