Under consideration for publication in Math. Struct. in Comp. Science

De Groot Duality and Models of Choice: Angels, Demons, and Nature

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Received February 2009

We introduce convex-concave duality for various models of non-deterministic choice, probabilistic choice, and the two of them together. This complements the well-known duality of stably compact spaces in a pleasing way: convex-concave duality swaps angelic and demonic choice, and leaves probabilistic choice invariant.

1. Introduction

There is a well-known duality on stably compact spaces X called de Groot duality: the de Groot dual X^{d} of X is X with its cocompact topology, and its own dual gives back X. We show that this induces another duality, or rather another family of dualities, on semantic models of choice, which swaps angels and demons, but leaves nature invariant.

What we mean by this is as follows. We show in Section 3 that the dual of the Smyth powerdomain (for *demonic* non-determinism) of any stably compact space X is the Hoare powerdomain (for *angelic* non-determinism) of the dual X^{d} of X, and conversely. It follows easily that both powerdomains are stably compact. Similarly, we show in Section 4 that the Plotkin powerdomain construction is self-dual: the dual of the Plotkin powerdomain (for erratic non-determinism) is the Plotkin powerdomain of the dual. We show it again in Section 5, using a functional view of powerdomains. This was our original approach. The functional view will be the one that prevails in the subsequent sections, and Section 5 will therefore serve as a, hopefully gentle, introduction.

In Section 6 we turn to the probabilistic powerdomain (Jones, 1990), which encodes probabilistic choice, and show that the probabilistic powerdomain construction is also self-dual. In a poetic way, Papadimitriou used the term “nature” to denote random choice (Papadimitriou, 1985), whence our claim that duality leaves nature invariant. One can vindicate this term as follows. In natural sciences, e.g., physics, phenomena are not thought as the result of a malevolent attacker (a demon) or of a benevolent angel, but as the by-product of mechanisms governed purely by probability distributions.

† Work partially supported by the INRIA ARC ProNoBiS.

We then turn to two models of our own that mix non-deterministic choice and probabilistic choice: the *game* models, an elaboration of Choquet's theory of capacities (Choquet, 54) with domain-theoretic flavor, which we deal with in Section 6 as well, since much can be factored with the purely probabilistic case; and the *prevision* models, in Section 7, which can either be seen as a similar elaboration from Walley's previsions (Walley, 1991), or as a relaxation of the game models through integral representation theorems. The prevision models are isomorphic to Tix, Keimel and Plotkin's own models of convex powercones (Tix et al., 2005), see (Goubault-Larrecq, 2008a; Keimel and Plotkin, 2009). In any case, we show similar duality theorems, which exchange demonic and angelic non-determinism while keeping probabilistic choice (nature) invariant.

2. Preliminaries

We refer the reader to (Abramsky and Jung, 1994; Gierz et al., 2003; Mislove, 1998) for background material on domain theory and topology, and recall some prerequisites in topology first, then on de Groot duality, and finally on domain theory.

2.1. *Topology*

A *topology* on X is a family of subsets, called the *opens*, such that any union and any finite intersection of opens is open. The complements of open subsets are called *closed*. The largest open contained in a subset A of X is its *interior* $int(A)$, while the smallest closed set containing A is its *closure* $cl(A)$. Given a subset A of X, the *induced topology* on A has as opens the intersections $A \cap U$, U open in X. A topology is *finer* than another iff it has at least as many opens.

Given any family of subsets $\mathcal{A}$ of X, there is a smallest topology on X *generated by* $\mathcal{A}$, i.e., making all elements of $\mathcal{A}$ open. Then every open in this topology is a union of finite intersections of elements of $\mathcal{A}$; $\mathcal{A}$ is then a *subbase* of the topology. If every open is a union of elements of $\mathcal{A}$, then $\mathcal{A}$ is called a *base* of the topology. We shall call *subbasic opens* the elements of a given subbase, and similarly for *basic opens*. By analogy, let the *subbasic closed sets* be the complements of the subbasic opens.

A map $f : X \to Y$ is *continuous* iff $f^{-1}(U)$ is open in X for every open subset U of Y. We shall often use the fact that, if $\mathcal{A}$ is a subbase of the topology of Y, $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open in X for all elements U of $\mathcal{A}$.

We reserve the term *homeomorphism* for isomorphisms in the category of topological spaces and continuous maps, i.e., one-to-one continuous maps whose inverse is also continuous. An *embedding* $f : X \to Y$ is a homeomorphism onto its image $\mathrm{Im}\, f$, i.e., an injective continuous map such that, for every open subset U of X, $f(U)$ is open in $\mathrm{Im}\, f$ — meaning that $f(U)$ can be written as $\mathrm{Im}\, f \cap V$ for some open subset V of Y.

The *product* $\prod_{i \in I} X_i$ of topological spaces is defined as the set-theoretic product, with the *product topology*, i.e., the smallest that makes all canonical projections $\pi_i : \prod_{i \in I} X_i \to X_i$ continuous. Alternatively, a base of this topology consists of products $\prod_{i \in I} U_i$ of opens U_i of X_i, $i \in I$, where only finitely many such opens U_i are different from X_i.

A subset Q of X is *compact* iff one can extract a finite subcover from every open cover of Q. It is *saturated* iff it is the intersection of all opens containing it, a.k.a. it is upward-closed in the *specialization quasi-ordering* $\leq$, defined by $x \leq y$ iff every open containing x contains y. The *saturation* $\uparrow A$ of a subset A of X is defined equivalently as the intersection of all opens U containing A, or as the upward-closure $\{x \in X \mid \exists y \in A \cdot y \leq x\}$. We write $\downarrow A$ the downward-closure $\{x \in X \mid \exists y \in A \cdot x \leq y\}$. Every open subset is upward-closed, and every closed subset is downward-closed. In T_0 spaces X that are not T_1, such as dcpos (see below), there are compact subsets that are not saturated, e.g., $\{x\}$ where x is not maximal in X. However, for any compact subset K, $\uparrow K$ is both compact and saturated.

A useful trick is *Alexander's Subbase Lemma* (Kelley, 1955, Theorem 5.6), which states that in a space X with subbase $\mathcal{A}$, a subset K is compact if and only if one can extract a finite subcover from every cover of K consisting of elements of $\mathcal{A}$.

A topological space X is *stably compact*—taking the definitions of (Jung, 2004; Alvarez-Manilla et al., 2004)—iff X is T_0 ($\leq$ is an ordering), *well-filtered* (for every filtered family $(Q_i)_{i \in I}$ of compact saturated subsets, for every open U, if $\bigcap_{i \in I} Q_i \subseteq U$ then $Q_i \subseteq U$ already for some $i \in I$), *locally compact* (whenever $x \in U$ with U open, there is a compact saturated subset Q such that $x \in int(Q) \subseteq Q \subseteq U$), *coherent* (the intersection of any two compact saturated subsets is again so) and *compact*. Stable compactness has a long history, going back to (Nachbin, 1948), see also (Jung, 2004; Alvarez-Manilla et al., 2004).

Be aware that several authors define "coherent" as synonymous with stably compact, leaving coherence itself without a name. In defining coherence as above, we follow, e.g., Alvarez-Manilla (Alvarez-Manilla, 2000). Note also that, in any locally compact space, whenever Q is a compact saturated subset of some open U, then there is a compact saturated subset Q_1 such that $Q \subseteq int(Q_1) \subseteq Q_1 \subseteq U$. In particular, every open is the union of the interiors of its compact saturated subsets.

Stable compactness is also usually defined by requiring sobriety instead of well-filteredness. As remarked by Jung (Jung, 2004, Section 2.3), referring to (Gierz et al., 2003, Theorem II-1.21), this is equivalent in the presence of local compactness. We shall only rarely need to refer to sobriety, however let us recall the definition. Say that a closed subset F of X is *irreducible* if and only if F is non-empty, and whenever F is contained in the union of two closed subsets F_1 and F_2, then F is contained in one of them already. Note that $\downarrow x$ is closed and irreducible for every element x of X. A space X is *sober* if and only if it is T_0, and the only irreducible closed sets are of the form $\downarrow x$, $x \in X$. The fundamental theorem of sober spaces is the Hofmann-Mislove Theorem (Abramsky and Jung, 1994, Theorem 7.2.9), which states that in a sober space X, the space of compact saturated subsets of X ordered by reverse inclusion, and the space of Scott-open filters of open subsets of X ordered by inclusion, are isomorphic. A *filter* $\mathcal{F}$ of opens of X is a family of open subsets of X containing X itself, such that the intersection of any two elements of $\mathcal{F}$ is again in $\mathcal{F}$, and any open V containing $U \in \mathcal{F}$ is in $\mathcal{F}$. $\mathcal{F}$ is *Scott-open* iff for every family $(U_i)_{i \in I}$ of open subsets of X whose union is in $\mathcal{F}$, some finite union $\bigcup_{i \in J} U_i$ (J finite subset of I) is already in $\mathcal{F}$; i.e., iff $\mathcal{F}$ is open in the lattice $\mathcal{O}(X)$ of open subsets of X with its Scott topology (see below). The easy direction consists in observing that, for every compact saturated subset Q of X, the collection of all opens containing

Q is a Scott-open filter of opens of X. Conversely, the intersection $Q = \bigcap_{U \in \mathcal{F}} U$ of all elements of a Scott-open filter $\mathcal{F}$ of opens is compact saturated, and the opens containing Q are exactly the elements of $\mathcal{F}$. It follows that, even without local compactness, sobriety implies well-filteredness.

Finally, we define stably *locally* compact spaces X as those obeying all the properties of stably compact spaces except, possibly, compactness. I.e., stably locally compact spaces are T_0, well-filtered, locally compact coherent spaces. Every stably locally compact space X can be embedded into a stably compact space $X_\perp$: $X_\perp$ is X plus a fresh element $\perp$, and the opens of $X_\perp$ are those of X plus $X_\perp$ itself. Note that $\perp$ is then the least element of $X_\perp$ in its specialization ordering.

A typical example of a stably compact space is the set $[0, 1]$ with opens of the form $(t, 1]$, $0 \leq t \leq 1$, plus $[0, 1]$ itself. This is just $[0, 1]$ with the Scott topology of its natural ordering $\leq$, see below. We shall write $[0, 1]_\sigma$ for $[0, 1]$ with its Scott topology, reserving $[0, 1]$ for (the set) $[0, 1]$ with its usual, metric topology. Similarly, we write $\mathbb{R}^+_\sigma$ for $\mathbb{R}^+$ with its Scott topology, whose non-trivial opens are the open intervals $(t, +\infty)$, $t \in \mathbb{R}^+$.

A space is *Hausdorff*, or T_2, iff every two distinct points x, y can be separated by opens U, V, i.e., $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Every compact T_2 space is stably compact, e.g., $[0, 1]$; the converse fails, as for example $[0, 1]_\sigma$ is stably compact but not T_2.

2.2. *De Groot Duality*

De Groot duality is the duality that the title of this paper refers to. My preferred reference to this theory is Jung's paper (Jung, 2004), and its journal version (Alvarez-Manilla et al., 2004). The study of compact pospaces originates in Nachbin's classic book (Nachbin, 1965). See also (Gierz et al., 2003, Section VI-6).

A *compact pospace* is a pair $(X', \preceq)$ where X' is a compact space and $\preceq$ is an ordering on X' whose graph is closed in $X' \times X'$. It follows that X' is compact T_2. The collection of opens of X' that are upward-closed in $\preceq$ forms a topology, the *upper topology*, which makes X' a stably compact space. Similarly, the *lower topology* consists in the downward-closed opens of X'.

For example, the upper topology of $([0, 1], \leq)$ is $[0, 1]_\sigma$, and its lower topology is given by subsets of the form $[0, t)$, $0 \leq t \leq 1$, plus $[0, 1]$ itself.

Nicely enough, one can go back, and retrieve a compact pospace from any stably compact space X. The first step consists in building the *cocompact topology* of X, which is the one generated by the *cocompact* subsets of X—the cocompact subsets are the complements $X \setminus Q$ of compact saturated subsets Q. The space X^{d} whose underlying set is that of X, and whose topology is the cocompact topology of X, is the *de Groot dual* of X. When X is stably compact, the opens of X^{d} are exactly the cocompacts of X. Note that this means that the closed subsets of X^{d} are the compact saturated subsets of X, and in particular that finite unions and arbitrary intersections of compact saturated subsets of X are again compact saturated.

Then, $X^{\mathsf{dd}} = X$; we take $=$ to denote equality of spaces, meaning that the two spaces have the same elements, and the same topologies as well. Also, the specialization ordering of X^{d} is $\geq$, the converse of the specialization ordering $\leq$ of X.

For example, when $X = [0,1]_\sigma$, the compact saturated subsets of X are the closed subintervals $[t,1]$, $0 \leq t \leq 1$, so the opens X^{d} are the subsets of the form $[0,t)$, $0 \leq t \leq 1$, plus $[0,1]$ itself—i.e., those of the lower topology of $[0,1]$.

The *patch topology* on X is generated by the union of the original topology of X and of its cocompact topology. Write X^{patch} for X with its patch topology. When X is stably compact, with specialization ordering $\leq$, $(X^{\mathsf{patch}}, \leq)$ is a compact pospace, which we shall call the *Nachbin pospace* of X. Its upper topology is the topology of X, and its lower topology is the cocompact topology of X. Conversely, if $(X', \preceq)$ is a compact pospace, then $(X', \preceq)$ is the Nachbin pospace of the space X' with its upper topology. E.g., the patch topology of $[0,1]_\sigma$ is the usual metric topology on $[0,1]$.

This machinery allows one to show easily that every patch-closed subset of a stably compact space X is stably compact in its induced topology (Jung, 2004, Proposition 2.16). A subset is *patch-closed* iff it is closed in the patch topology. This generalizes the fact that every closed subset of a compact T_2 space is compact T_2, to non-T_2 topologies.

Moreover, these constructions behave well with respect to products. For every family $(X_i)_{i \in I}$ of stably compact spaces, where $\leq_i$ is the specialization ordering of X_i, $i \in I$, then $X' = \prod_{i \in I} X_i^{\mathsf{patch}}$ with the componentwise ordering $\leq$ is a compact pospace, and is in fact the Nachbin pospace of $\prod_{i \in I} X_i$. In other words, the patch topology of a product of stably compact spaces is the product of the patch topologies, and the specialization ordering of the product is the componentwise ordering (Jung, 2004, Proposition 2.15).

For example, the space $[0,1]_\sigma^I$ of all maps from I to $[0,1]_\sigma$, seen as the product of I copies of $[0,1]_\sigma$, can also be seen as the product of I copies of $[0,1]$, with the upper topology of the componentwise ordering. We call such spaces $[0,1]_\sigma^I$ *cubes*.

2.3. *Domain Theory*

A set with a partial ordering is a *poset*. A *dcpo* is a poset in which every directed family $(x_i)_{i \in I}$ has a least upper bound (a.k.a., *sup*) $\sup_{i \in I} x_i$. A family $(x_i)_{i \in I}$ is *directed* iff it is non-empty, and any two elements have an upper bound in the family. Any poset can be equipped with the *Scott topology*, whose opens are the upward closed sets U such that whenever $(x_i)_{i \in I}$ is a directed family that has a least upper bound in U, then some x_i is in U already. The Scott topology is always T_0, and its specialization ordering is the original partial ordering.

The *way-below* relation $\ll$ on a poset X is defined by $x \ll y$ iff, for every directed family $(z_i)_{i \in I}$ that has a least upper bound z such that $y \leq z$, then $x \leq z_i$ for some $i \in I$ already. Note that $x \ll y$ implies $x \leq y$, and that $x' \leq x \ll y \leq y'$ implies $x' \ll y'$. However, $\ll$ is not reflexive or irreflexive in general. Write $\twoheaduparrow E = \{y \in X \mid \exists x \in E \cdot x \ll y\}$, $\twoheaddownarrow E = \{y \in X \mid \exists x \in E \cdot y \ll x\}$. X is *continuous* iff, for every $x \in X$, $\twoheaddownarrow x$ is a directed family, and has x as least upper bound. One may be more precise: A *basis* is a subset B of X such that any element $x \in X$ is the least upper bound of a directed family of elements way-below x *in* B. Then X is continuous if and only if it has a basis, and in this case X itself is the largest basis. In a continuous poset X with basis B, the *interpolation property* holds: whenever $x \ll z$, then $x \ll y \ll z$ for some $y \in B$ (Mislove, 1998, Lemma 4.16). It follows that, in any continuous poset X, $\twoheaduparrow x$ is Scott-open for all x, and every Scott-open set U is a union of such sets, more precisely $U = \bigcup_{x \in U \cap B} \twoheaduparrow x$.

In any topological space, call *finitary compact* any subset of the form $\uparrow E$ with E finite. Every finitary compact is compact saturated. In a continuous dcpo X, the compact saturated subsets Q are exactly the intersections $\bigcap_{i \in I} \uparrow E_i$ of filtered families $(\uparrow E_i)_{i \in I}$ of finitary compacts, see e.g., (Jung, 1998, Lemma 4.10, *(ii)*). (We order subsets by inclusion, and a family is *filtered* if and only if it is directed in the converse ordering.) It follows easily that the topology generated by the complements of sets $\uparrow x$, $x \in X$ (the *upper topology* of the ordering $\leq$), on a continuous dcpo X, is exactly the cocompact topology (Lawson, 1988, Section V), so the patch topology on X coincides with the Lawson topology. The latter is probably more well-known in domain theory, and is the one generated by the Scott opens and the complements of sets $\uparrow x$, $x \in X$.

Unless otherwise mentioned, we shall always see every dcpo as a topological space, with the Scott topology of its ordering. Every continuous dcpo X is sober hence well-filtered, and locally compact. If additionally X is *pointed*, i.e., has a least element $\bot$, then X is compact. If finally X is also coherent, then X is stably compact. The reader may be more familiar with the notion of Lawson-compactness, where X is *Lawson-compact* iff it is compact in its Lawson topology. By the discussion above, Lawson-compactness is equivalent to stable compactness on continuous dcpos. This is in fact true even on quasi-continuous dcpos (Gierz et al., 2003, Theorem III-5.8), although it may fail on more general classes of dcpos.

Note that $[0, 1]_\sigma$ is a continuous pointed dcpo, where $x \ll y$ if and only if $x = 0$ or $x < y$. As we have seen, it is also stably compact. For any set I, then, $[0, 1]_\sigma^I$ is also a stably compact, continuous, pointed dcpo. One can check that $\ll$ is defined by $f \ll g$ iff $f(i) = 0$ for all i except for those in some finite subset J of I, and $f(i) < g(i)$ for all $i \in J$.

Bc-domains are bounded-complete continuous dcpos; *bounded-completeness* means that any two elements x, y that have an upper bound also have a least upper bound. Any bc-domain is coherent, hence stably locally compact. Any pointed bc-domain is stably compact.

Given any poset X, with partial order $\leq$, the *opposite* poset X^{op} has the same elements as X, and its ordering is the converse $\geq$ of $\leq$. We clearly have $X^{\mathsf{op}\ \mathsf{op}} = X$, where equality is equality of posets, which implies the equality of orderings, hence also of the Scott topologies.

A poset X is a *bicpo* if and only if both X and X^{op} are dcpos, meaning that directed families have sups, and that filtered families have infs. Any stably compact space is a bicpo for its specialization ordering: this follows easily from the fact that any sober space is a dcpo in its specialization ordering (Abramsky and Jung, 1994, Proposition 7.2.13), applied to X and X^{d}.

A *bicontinuous bicpo* is a poset X such that both X and X^{op} are continuous dcpos. In this case, we still write $\ll$ for the way-below relation of X, and we write $\gg$ for the way-below relation of X^{op}. Call $\gg$ the *way-above* relation of X: $y \gg x$ if and only if, for every filtered family $(z_i)_{i \in I}$ that has a greatest lower bound (a.k.a., an *inf*) z such that $z \leq x$, then $z_i \leq y$ for some $i \in I$ already. Beware that in general $\gg$ is not the converse of $\ll$. As an example, $[0,1]$ with its natural ordering is a bicontinuous bicpo. However, $x \ll y$ iff $x = 0$ or $x < y$, while $y \gg x$ iff $y = 1$ or $x < y$; so $0 \ll 0$ but $0 \not\gg 0$, while $1 \not\ll 1$ and $1 \gg 1$.

3. Powerdomains for Non-Deterministic Choice: The One-Sided Cases

Fix a state space X. Modeling non-deterministic choice from a subset of elements A of X is simply done by specifying A, an element of the powerset $\mathbb{P}(X)$. When instead X is a dcpo, or in fact any topological space, one has to replace $\mathbb{P}(X)$ by an appropriate notion of *powerdomain*. Then several notions of powerdomains arise. We refer again to (Abramsky and Jung, 1994; Gierz et al., 2003; Mislove, 1998) for background on powerdomains.

So, in the general case, fix a topological space X.

First, there is *demonic* choice, best modeled through the *Smyth powerdomain* $\mathcal{Q}(X)$. Its elements are the non-empty compact saturated subsets Q of X. One domain-theoretic tradition is to see $\mathcal{Q}(X)$ as a poset, with reverse inclusion $\supseteq$. When X is well-filtered, this yields a dcpo, and $\sup_{i \in I} Q_i = \bigcap_{i \in I} Q_i$ for any directed (i.e., filtered for $\subseteq$) family $(Q_i)_{i \in I}$. If X is also locally compact, then this dcpo is also continuous, and $Q \ll Q'$ iff $int(Q) \supseteq Q'$. When X is coherent, $\mathcal{Q}(X)$ is bounded-complete, Q and Q' have an upper bound if and only if $Q \cap Q'$ is non-empty, and their least upper bound is $Q \cap Q'$. Finally, when X is compact, then $\mathcal{Q}(X)$ is pointed, and the least element is X itself. In particular, $\mathcal{Q}(X)$ is a stably compact, continuous, pointed dcpo for any stably compact space X.

Another, more topological, definition of a topology on $\mathcal{Q}(X)$ is the *upper Vietoris* topology, which has a base given by subsets of the form $\Box U = \{Q \in \mathcal{Q}(X) \mid Q \subseteq U\}$, U open in X. It is easy to see that every open in this topology is Scott-open, while if X is well-filtered and locally compact, $\mathop{\Uparrow} Q = \Box int(Q)$. Hence, in this case, the two topologies coincide. Since we shall focus on stably compact spaces X, we shall therefore switch freely between the two. Note also that the upper Vietoris topology is generated from subsets of the form $\Box int(Q)$, Q compact saturated in X.

There are several ways to explain why elements of $\mathcal{Q}(X)$ model *demonic* choice. A deep one is to show that $\mathcal{Q}(X)$ is the free dcpo-algebra of an inequational theory involving one associative, commutative and idempotent symbol $\uplus$ that must obey the extra inequality $x \uplus y \sqsubseteq x$ (Abramsky and Jung, 1994, Section 6.2.2). This holds as soon as X is a continuous dcpo; in $\mathcal{Q}(X)$, $\uplus$ is then just union.

I tend to prefer the following. First, if X is a continuous dcpo, any element of $\mathcal{Q}(X)$ is the directed sup of finitary compacts $\uparrow E$. Then, if choice is described by $\uparrow E$, the elements of $\uparrow E$ are all possible choices, and the elements of E are the *worst* (least) possible ones. A demon (think: a malicious scheduler that is required to pick one of the possible choices) is then specified by what worst possible choices it could make. Since E is finite, this explanation is only relevant for so-called finitely branching systems, see e.g. (Johnstone et al., 1998) and references therein. I'll propose a better explanation in Section 7, see Lemma 7.5 and subsequent discussion.

The second form of choice is *angelic* choice, which is modeled using the *Hoare powerdomain* $\mathcal{H}(X)$. Its elements are the non-empty closed subsets of X There is again a domain-theoretic and a topological definition of the topology one puts on $\mathcal{H}(X)$, but they tend to differ in more cases than for the Smyth powerdomain.

The domain-theoretic definition is to order $\mathcal{H}(X)$ by inclusion $\subseteq$. This is a complete lattice, hence certainly a dcpo (whatever X is). The sup of the directed family $(F_i)_{i\in I}$ is the *closure* $cl(\bigcup_{i\in I} F_i)$ of the union. However, $\mathcal{H}(X)$ is known to be a continuous dcpo only when X is itself a continuous dcpo, or at least a continuous poset (one can e.g., adapt Theorem 4.23 of (Mislove, 1998) in the latter case); then the way-below relation on $\mathcal{H}(X)$ is defined by $F \ll F'$ iff there is a non-empty finite subset E of (a given basis of) X such that $F \subseteq \downarrow E$ and $E \subseteq \Downarrow F'$. This is easily deduced from (Abramsky and Jung, 1994, Theorem 6.2.10, Theorem 6.2.13). Then the subsets $\downarrow E$ themselves, with E non-empty and finite subsets of a given basis of X, form a basis of $\mathcal{H}(X)$, and $\downarrow E \ll F'$ in $\mathcal{H}(X)$ iff $E \subseteq \Downarrow F'$.

The topological definition is to equip $\mathcal{H}(X)$ with its *lower Vietoris* topology, generated by subbasic opens $\Diamond U = \{F \in \mathcal{H}(X) \mid F \cap U \neq \emptyset\}$, U open in X. We shall write $\mathcal{H}_{\mathcal{V}}(X)$ for this topological space, to distinguish it from the dcpo $\mathcal{H}(X)$. Note that, letting F be the complement of U, the complement of $\Diamond U$ is $\blacksquare F = \{F' \in \mathcal{H}(X) \mid F' \subseteq F\}$, i.e., the downward-closure of $\{F\}$ in $\mathcal{H}(X)$.

It is easy to see that $\Diamond F$ is Scott-open, so the topology of $\mathcal{H}(X)$ is finer than that of $\mathcal{H}_{\mathcal{V}}(X)$. When X is a continuous dcpo, the two topologies coincide, i.e., $\mathcal{H}_{\mathcal{V}}(X) = \mathcal{H}(X)$, because, conversely, any Scott open is a union of sets of the form $\Uparrow_{\mathcal{H}(X)}(\downarrow_X E)$ (subscripts indicate in which spaces we must take various arrows), i.e., of the form $\{F' \in \mathcal{H}(X) \mid E \subseteq \Downarrow F'\} = \bigcap_{x\in E} \Diamond \Uparrow x$.

In case X is a continuous dcpo, $\mathcal{H}(X)$ can be described as a similar free dcpo-algebra, this time using the converse inequality $x \sqsubseteq x \uplus y$; $\uplus$ again denotes union. One can also see the elements $\downarrow E$ of the above basis for $\mathcal{H}(X)$ as specified by the set E of *best* (highest) possible choices.

We will turn to the Plotkin powerdomain in Section 4. However, we observe right away that, in the world of stably compact spaces, Smyth and Hoare are dual of each other. Theorem 3.1 is the first trace we observe of the action of de Groot duality on models of choice.

Theorem 3.1 (Duality, One-Sided Non-Deterministic Case). Let X be a stably compact space. Then $\mathcal{Q}(X)$ is stably compact, and $\mathcal{Q}(X)^{\mathsf{d}} = \mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$.

Proof. First, we have already noticed that $\mathcal{Q}(X)$ is stably compact. Then, $\mathcal{Q}(X)^{\mathsf{d}}$ and $\mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$ certainly have the same elements, namely the non-empty compact saturated subsets of X. We must now show that their topologies coincide, and we do this by showing that the two spaces have the same closed subsets.

Since X is well-filtered and locally compact, $\mathcal{Q}(X)$ is not only stably compact, but also a continuous dcpo. So $\mathcal{Q}(X)$ is Lawson-compact, and the cocompact and upper topologies of its ordering $\supseteq$ coincide. This means that the upward-closures of elements Q of $\mathcal{Q}(X)$ form a subbase of closed sets for $\mathcal{Q}(X)^{\mathsf{d}}$. These upward-closures are just $\blacksquare Q$. Conversely, we have already noted, in introducing the lower Vietoris topology, that a subbase of closed sets for $\mathcal{H}(X^{\mathsf{d}})$ is given by all sets of the form $\blacksquare F$, F closed in X^{d}. Since $\mathcal{Q}(X)^{\mathsf{d}}$ and $\mathcal{H}(X^{\mathsf{d}})$ have the same subbasic closed sets, their topologies agree. □

Corollary 3.2. For any stably compact space X, $\mathcal{Q}(X)$ and $\mathcal{H}_{\mathcal{V}}(X)$ are stably compact.

Indeed, by Theorem 3.1, $\mathcal{H}_{\mathcal{V}}(X) = \mathcal{Q}(X^{\mathsf{d}})^{\mathsf{d}}$. That $\mathcal{H}_{\mathcal{V}}(X)$ is stably compact when X is was proved by Cohen (Cohen, 2006), although with a more complex argument.

Another easy consequence of Theorem 3.1 is more domain-theoretic:

Corollary 3.3. Let X be a stably compact continuous dcpo. Then $\mathcal{H}(X)$ is a stably compact, bicontinuous bicpo, and is (qua poset) the opposite of $\mathcal{Q}(X^{\mathsf{d}})$.

Proof. Since X is a continuous dcpo, $\mathcal{H}(X)$ is, too. Also, $\mathcal{H}(X) = \mathcal{H}_{\mathcal{V}}(X)$. By Theorem 3.1, $\mathcal{H}_{\mathcal{V}}(X) = \mathcal{Q}(X^{\mathsf{d}})^{\mathsf{d}}$. Looking at the underlying posets, $\mathcal{H}(X)$ equals $\mathcal{Q}(X^{\mathsf{d}})^{\mathsf{op}}$. In particular, the opposite $\mathcal{H}(X)^{\mathsf{op}}$ equals $\mathcal{Q}(X^{\mathsf{d}})$, which is a continuous dcpo, since X^{d} is well-filtered and locally compact. □

In general, the way-below and the converse of the way-above relations do not coincide on $\mathcal{H}(X)$. Indeed, take $X = [0,1]_\sigma$. The elements of $\mathcal{H}([0,1]_\sigma)$ are of the form $[0,t]$, $0 \leq t \leq 1$, so that $\mathcal{H}([0,1]_\sigma)$ is canonically isomorphic to $[0,1]_\sigma$; but we have seen that the way-below and converse of the way-above did not coincide on the latter.

The first part of Corollary 3.3, that $\mathcal{H}(X)$ is a stably compact, bicontinuous bicpo, is actually an instance of a more general result, which does not require stable compactness. If X is a continuous dcpo, then $\mathcal{O}(X)$ is a completely distributive lattice (Gierz et al., 2003, Theorem II-1.14). It follows that its opposite $\mathcal{H}_\perp(X)$, the poset of all closed subsets (including the empty set), is also a completely distributive lattice. This is more than what we claimed above for $\mathcal{H}(X)$: any completely distributive lattice is a bicontinuous bicpo (Gierz et al., 2003, Theorem I-3.16(2)), and is compact T_2 in its Lawson topology (Gierz et al., 2003, Corollary III-1.11), hence stably compact. In fact, completely distributive lattices L have many other properties: the Scott and upper topologies coincide on L; the Scott topology of L^{op} then coincides with the lower topology, generated by complements of set of the form $\uparrow x$, $x \in L$; the Lawson topologies of L and L^{op} both coincide with the interval topology, generated by complements of intervals $\uparrow x \cap \downarrow y$, so that L is *linked bicontinuous* (Gierz et al., 2003, Proposition VII-2.10).

$\mathcal{H}(X)$ need not form a lattice except when X has a bottom element. However, we can still show that $\mathcal{H}(X)$ is a stably compact, bicontinuous bicpo provided X is a compact continuous dcpo. Since X is continuous, $\mathcal{H}(X)$ is, too. Then $\mathcal{H}(X)$ is a bc-domain, and is therefore stably compact and continuous. $\mathcal{H}(X)^{\mathsf{op}}$ is isomorphic to the poset $\mathcal{O}^*(X)$ of all open subsets of X distinct from X. We shall argue in Lemma 3.8 that, if X is compact, then $\mathcal{O}^*(X)$ is a bc-domain. In particular, $\mathcal{H}(X)^{\mathsf{op}}$ is also a bc-domain, hence also stably compact and continuous. This only requires X to be a compact, not stably compact, continuous dcpo.

Let us turn to the Smyth powerdomain. We agree to say that X^{d} is a continuous dcpo iff the specialization ordering $\geq$ of X^{d} makes X^{d} a continuous dcpo, *and* that the topology of X^{d} coincides with the Scott topology. This accords with our convention that equality of topological spaces implies equality of their topologies.

Corollary 3.4. Let X be a stably compact space whose de Groot dual X^{d} is a continuous dcpo. Then $\mathcal{Q}(X)$ is a stably compact, bicontinuous bicpo, and is (qua poset) the opposite of $\mathcal{H}(X^{\mathsf{d}})$.

Proof. Since X is well-filtered and locally compact, $\mathcal{Q}(X)$ is a continuous dcpo. Since X^{d} is a continuous dcpo, $\mathcal{H}(X^{\mathsf{d}}) = \mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$ is, too. Then, as a poset, $\mathcal{Q}(X)$ is the opposite of $\mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$ by Theorem 3.1. So $\mathcal{Q}(X)$ is a bicontinuous bicpo. □

Note that, similarly to $\mathcal{H}_{\perp}(X)$ discussed above, $\mathcal{Q}^{\top}(X)$, defined as the collection of all compact saturated subsets of X, including the empty set, is again a completely distributive lattice in this case, which is much more than a stably compact, bicontinuous bicpo. This is because $\mathcal{Q}^{\top}(X)$ is the lattice of complements of elements of $\mathcal{O}(X^{\mathsf{d}})$.

We obtain a completely symmetric result by using the following class of spaces, which occurs naturally in this context.

Definition 3.5. A *stably bicontinuous bicpo* is a stably compact space X such that both X and X^{d} are continuous dcpos.

By this we again mean that the topologies of X and of X^{d} are the Scott topologies of the underlying posets X and X^{op}, and that these posets are continuous dcpos. Any cube $[0,1]^I_\sigma$, for any set I, is a stably bicontinuous bicpo. In fact, the cubes are completely distributive lattices, and any completely distributive lattice is a stably bicontinuous bicpo.

We have seen that, on the cube, $f \ll g$ iff $f(i) = 0$ for all i except for those in some finite subset J of I, and $f(i) < g(i)$ for all $i \in J$. Symmetrically, $g \gg f$ iff $g(i) = 1$ for all i except for those in some finite subset J of I, and $f(i) < g(i)$ for all $i \in J$. So $\ll$ is not the opposite of $\gg$ in general.

In a stably bicontinuous bicpo, it is equivalent to talk about X^{d} or X^{op}: X^{op} is indeed a stably compact continuous dcpo that coincides with X^{d}. In fact, one may also define stably bicontinuous bicpos as bicontinuous dcpos X such that X and X^{op} have the same Lawson (or patch) topology. E.g., the Lawson topology on $[0,1]^I_\sigma$, as well as on $([0,1]^{\mathsf{op}})^I_\sigma$ is the usual product topology on $[0,1]^I$.

Corollary 3.6. For any stably bicontinuous bicpo X, $\mathcal{Q}(X)$ and $\mathcal{H}(X)$ are stably bicontinuous bicpos, and $\mathcal{Q}(X)^{\mathsf{op}} = \mathcal{H}(X^{\mathsf{op}})$, $\mathcal{H}(X)^{\mathsf{op}} = \mathcal{Q}(X^{\mathsf{op}})$.

This follows directly from Corollaries 3.3 and 3.4.

Another corollary of Theorem 3.1 is as follows.

Lemma 3.7. Let X be stably compact. For every compact saturated subset Q of X, $\blacksquare Q = \{Q' \in \mathcal{Q}(X) \mid Q' \subseteq Q\}$ is compact saturated in $\mathcal{Q}(X)$.

This can also be proved independently, using Alexander's Subbase Lemma. In fact, another proof of Theorem 3.1 consists in showing that any subbasic closed set for one topology is closed in the other one, and starts from Lemma 3.7: note that $\blacksquare Q$ is a subbasic closed set in $\mathcal{H}(X^{\mathsf{d}})$, and is compact saturated in $\mathcal{Q}(X)$ by Lemma 3.7; conversely, since $\mathcal{Q}(X)$ is a continuous dcpo, every compact saturated subset of $\mathcal{Q}(X)$ is a directed union of finitary compacts $\uparrow_{\mathcal{Q}(X)} \mathcal{E}_i$, $i \in I$, where each $\mathcal{E}_i$ is a finite subset of $\mathcal{Q}(X)$, however $\uparrow_{\mathcal{Q}(X)} \mathcal{E}_i = \bigcup_{Q \in \mathcal{E}_i} \blacksquare Q$ is closed in $\mathcal{H}(X^{\mathsf{d}})$.

Theorem 3.1 has apparently not been published until now. However, Martín Escardó told me at the Domains IX Workshop in September 2008 that he had known about this result since 2000. He presented the following in a McGill university seminar in 2003, although this did not make it into the notes (Escardó, 2003). The following lemma is in fact how I also came to discover Theorem 3.1 (Goubault-Larrecq, 2007, Section 3.4); the proof using Lemma 3.7 above is a simplified argument; the even shorter argument given as proof for Theorem 3.1 is due to one of the anonymous referees. We let $\mathcal{O}(X)$ denote the complete lattice of all open subsets of X; recall that $\mathcal{O}^*(X) = \mathcal{O}(X) \setminus \{X\}$.

Lemma 3.8. Let X be a stably compact space. Then the complement map $A \mapsto X \setminus A$ defines a homeomorphism between:

$$\mathcal{H}_{\mathcal{V}}(X) \quad \text{and} \quad \mathcal{O}^*(X)^{\mathsf{d}}$$
$$\mathcal{Q}(X) \quad \text{and} \quad \mathcal{O}^*(X^{\mathsf{d}})$$

Proof. First, we observe that X is locally compact, hence $\mathcal{O}(X)$ is a continuous complete lattice, see e.g. Proposition 4.2.15 (Abramsky and Jung, 1994). Using similar arguments, one can see that $\mathcal{O}^*(X)$ is a continuous poset as soon as X is locally compact, where U is way-below V iff $U \subseteq Q \subseteq V$ for some compact saturated subset Q. $\mathcal{O}^*(X)$ is a dcpo provided X is compact: for any directed family of opens $(U_i)_{i \in I}$ in X, all distinct from X, their union cannot be X by compactness, and therefore serves as the sup of the family in $\mathcal{O}^*(X)$. It is then clear that $\mathcal{O}^*(X)$ is a bc-domain, hence is stably compact. So the notation $\mathcal{O}^*(X)^{\mathsf{d}}$ makes sense.

That the complements of elements of $\mathcal{H}_{\mathcal{V}}(X)$ are the elements of $\mathcal{O}^*(X)^{\mathsf{d}}$, i.e., the complements of open subsets of X other than X, is clear. Similarly, the complements of elements of $\mathcal{Q}(X)$ are the cocompact subsets of X distinct from X, i.e., the elements of $\mathcal{O}^*(X^{\mathsf{d}})$.

The topology of $\mathcal{O}^*(X)^{\mathsf{d}}$ is the cocompact topology of $\mathcal{O}^*(X)$, and coincides with the upper topology of the ordering, since $\mathcal{O}^*(X)$ is continuous. So it is generated by the complements of sets $\uparrow_{\mathcal{O}^*(X)} U$, U open in X, namely the sets $\{U' \in \mathcal{O}^*(X) \mid U \not\subseteq U'\} = \{X \setminus F \mid F \in \mathcal{H}(X), F \in \Diamond U\}$. The complement map sends this to $\Diamond U$. Conversely, the image in $\mathcal{O}^*(X)$ of $\Diamond U$ by the complement map is the complement of the finitary compact $\uparrow_{\mathcal{O}^*(X)} U$. So the complement map indeed defines a homeomorphism between $\mathcal{H}_{\mathcal{V}}(X)$ and $\mathcal{O}^*(X)^{\mathsf{d}}$.

The ordering on the image of $\mathcal{Q}(X)$ by the complement map is ordinary inclusion, the same as on $\mathcal{O}^*(X^{\mathsf{d}})$. Since the topologies on both spaces is entirely determined as the Scott topologies on the same ordering $\subseteq$, they are the same space. ∎

Theorem 3.1 easily follows. One can also observe that Lemma 3.8 implies that $\mathcal{Q}(\mathcal{H}_{\mathcal{V}}(X))$ is homeomorphic to $\mathcal{O}(\mathcal{O}(X))$, naturally in X, for every stably compact space X. Vickers and Townsend (Vickers and Townsend, 2004) observed that $\mathcal{H}_{\mathcal{V}}(\mathcal{Q}(X))$ is also homeomorphic to $\mathcal{O}(\mathcal{O}(X))$, naturally in X. The resulting space, up to homeomorphism is the *double powerdomain* of X. Vickers and Townsend reason in the category of locales instead of topological spaces, and call this the *double powerlocale*; they also don't require stable compactness.

4. Powerdomains for Erratic Choice I

The Plotkin powerdomain for erratic non-determinism has even more variants than the Hoare and Smyth powerdomains.

Let X be any space. A *lens* L of X is the intersection $Q \cap F$ of a compact saturated subset Q of X and a closed subset F of X, provided this intersection is non-empty. Then L has a canonical presentation as $\uparrow L \cap cl(L)$, where $\uparrow L$ is compact saturated, and $cl(L)$ is closed. There is a domain-theoretic definition, as a poset $\mathcal{P}\ell(X)$ of lenses, ordered by the *topological Egli-Milner ordering* $\sqsubseteq_{\mathrm{EM}}$, defined by $L \sqsubseteq_{\mathrm{EM}} L'$ iff $\uparrow L \supseteq \uparrow L'$ and $cl(L) \subseteq cl(L')$. When X is a stably compact continuous dcpo, then $\mathcal{P}\ell(X)$ is a stably compact continuous dcpo. Stable compactness was proved by Jimmie Lawson (Lawson, 1987, Theorem, p.156), see also (Mislove, 1998, Corollary 4.48). Continuity is claimed in (Abramsky and Jung, 1994, Exercise 6.2.23(11)), and follows from Construction IV.8.12 and subsequent theorems of (Gierz et al., 2003). A basis is given by the *finitary lenses*, i.e., the sets of the form $\langle E \rangle$, E non-empty finite, where $\langle E \rangle = \uparrow E \cap \downarrow E$. $\langle E \rangle$ is the set of elements that are above some minimal element of E (the *worst* choices) and below some maximal element of E (the *best* choices). Then $\langle E \rangle \ll L$ if and only if $E \subseteq \mathord{\downarrow\!\!\downarrow} F$ and $Q \subseteq \mathord{\uparrow\!\!\uparrow} E$, where $F = cl(L)$ and $Q = \uparrow L$.

There is also a topological version of the Plotkin powerdomain, which we shall write $\mathcal{P}\ell_{\mathcal{V}}(X)$, namely the space of lenses of X with the *Vietoris topology*, generated by sets $\{L \in \mathcal{P}\ell(X) \mid L \subseteq U\}$, which we shall write $\Box U$ again, and $\{L \in \mathcal{P}\ell(X) \mid L \cap U \neq \emptyset\}$, which we shall write $\Diamond U$, for any subset U of X. It is easy to see that the specialization ordering of $\mathcal{P}\ell_{\mathcal{V}}(X)$ is $\sqsubseteq_{\mathrm{EM}}$. The Scott topology of $\mathcal{P}\ell(X)$ is always finer than the Vietoris topology. When X is a stably compact, continuous dcpo, the converse holds, so $\mathcal{P}\ell_{\mathcal{V}}(X) = \mathcal{P}\ell(X)$. Indeed, any subbasic Scott-open $\mathord{\uparrow\!\!\uparrow}_{\mathcal{P}\ell(X)} \langle E \rangle$ can be written as the Vietoris open $\bigcap_{x \in E} \Diamond \mathord{\uparrow\!\!\uparrow} x \cap \Box \mathord{\uparrow\!\!\uparrow} E$.

Call a subset L of X *patch-compact* iff it is compact in X^{patch}, and *order-convex* iff whenever $x, z \in L$ and $x \leq y \leq z$, then $y \in L$. Clearly, every lens L is order-convex, and moreover if X is stably compact, then L is patch-closed, hence patch-compact in X. The converse also holds, see Lemma 4.2, which rests on the following fact.

Fact 4.1. Let X be a stably compact space. For every patch-compact subset L of X, $\downarrow L$ is closed in X, and $\uparrow L$ is compact saturated in X.

Indeed, L is compact both in X and in X^{d}, since the larger the topology, the smaller the collection of compact subsets. Then $\downarrow L$ is the saturation on L in X^{d}, since the specialization ordering of X^{d} is the opposite of $\leq$. It follows that $\downarrow L$ is compact saturated in X^{d}, hence closed in X. Similarly, $\uparrow L$ is compact saturated in X.

Lemma 4.2. Let X be stably compact. The lenses L of X are exactly the non-empty, patch-compact, order-convex subsets of X. Moreover, every lens is a *strong lens*, i.e., $\downarrow L = cl(L)$. Finally, the specialization ordering $\sqsubseteq_{\mathrm{EM}}$ on $\mathcal{P}\ell_{\mathcal{V}}(X)$ reduces to the ordinary Egli-Milner ordering, i.e., $L \sqsubseteq_{\mathrm{EM}} L'$ iff:

for every $x \in L$, there is an $x' \in L'$ such that $x \leq x'$, and
for every $x' \in L'$, there is an $x \in L$ such that $x \leq x'$.

Proof. We have given one direction of the proof above. Conversely, let L be non-empty, patch-compact, and order-convex. By Fact 4.1, $\downarrow L$ is closed, and $\uparrow L$ is compact saturated in X. Also, $L = \uparrow L \cap \downarrow L$ because L is order-convex, whence L is a lens. Moreover, since $\downarrow L$ is closed and contains L, it contains $cl(L)$. Conversely, $\downarrow L \subseteq cl(L)$ since every closed set is downward-closed. So $\downarrow L = cl(L)$, and theferore L is a strong lens.

The characterization of $\sqsubseteq_{\mathrm{EM}}$ follows from the fact that every lens is strong. $\square$

This is well-known when X is a stably compact continuous dcpo (Gierz et al., 2003, Proposition IV-8.17): in this case, the lenses are exactly the order-convex non-empty Lawson-compact subsets of X (Abramsky and Jung, 1994, Corollary 6.2.21), and we have already seen that Lawson-compactness and patch-compactness coincided on continuous dcpos.

Lemma 4.2 entails that $\mathcal{P}\ell_{\mathcal{V}}(X)^{\mathsf{d}}$ and $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$ have the same elements, namely the non-empty, patch-compact, order-convex subsets of X. Indeed, patch-compactness is the same notion in X and in X^{d}, while order-convexity is left invariant by reversing the ordering.

We now observe that the Plotkin powerdomain construction is *self-dual*, in the sense that $\mathcal{P}\ell_{\mathcal{V}}(X)^{\mathsf{d}}$ and $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$ not only have the same elements, but in fact also the same topology, hence are the same space. This will be Theorem 4.5, and is the second trace we observe of the action of de Groot duality on models of choice.

Lemma 4.3. In a topological space X, let:

$$\blacksquare Q = \{L \in \mathcal{P}\ell_{\mathcal{V}}(X) \mid L \subseteq Q\} \qquad \blacklozenge Q = \{L \in \mathcal{P}\ell_{\mathcal{V}}(X) \mid L \cap Q \neq \emptyset\}$$

for any compact saturated subset Q of X.

Let X be compact. Then $\blacksquare Q$ and $\blacklozenge Q$ are compact saturated in $\mathcal{P}\ell_{\mathcal{V}}(X)$, for any compact saturated subset Q.

Proof. Use Alexander's Subbase Lemma, and show that one can extract a finite subcover from a cover of $\blacksquare Q$ by subbasic opens $\Box U_i$, $i \in I$, and $\Diamond V_j$, $j \in J$. I.e., assume $\blacksquare Q \subseteq \bigcup_{i \in I} \Box U_i \cup \bigcup_{j \in J} \Diamond V_j$. Let $L_0 = Q \setminus \bigcup_{j \in J} V_j$.

If L_0 is empty, then $Q \subseteq \bigcup_{j \in J} V_j$, so $Q \subseteq \bigcup_{j \in J_0} V_j$ for some finite subset J_0 of J. Then $(\Diamond V_j)_{j \in J_0}$ is the desired finite subcover.

Otherwise, L_0 is a lens, $L_0 \in \blacksquare Q$, and $L_0 \notin \bigcup_{j \in J} \Diamond V_j$. So $L_0 \subseteq U_i$ for some $i \in I$. By the definition of L_0, then, $Q \subseteq U_i \cup \bigcup_{j \in J} V_j$, so $Q \subseteq U_i \cup \bigcup_{j \in J_0} V_j$ for some finite subset J_0 of J. Then $\Box U_i$ and $(\Diamond V_j)_{j \in J_0}$ form the desired finite subcover.

We show that $\blacklozenge Q$ is compact saturated in a similar way. Assume $\blacklozenge Q \subseteq \bigcup_{i \in I} \Box U_i \cup \bigcup_{j \in J} \Diamond V_j$. Let $L_0 = X \setminus \bigcup_{j \in J} V_j$.

If L_0 is empty, then $X \subseteq \bigcup_{j \in J} V_j$. Since X is compact, $X \subseteq \bigcup_{j \in J_0} V_j$ for some finite subset J_0 of J. Then $(\Diamond V_j)_{j \in J_0}$ is the desired finite subcover.

Otherwise, L_0 is a lens, again because X is compact, and $L_0 \notin \bigcup_{j \in J} \Diamond V_j$. So either $L_0 \notin \blacklozenge Q$ or $L_0 \subseteq U_i$ for some $i \in I$.

In the latter case, $X \subseteq U_i \cup \bigcup_{j \in J} V_j$, so $X \subseteq U_i \cup \bigcup_{j \in J_0} V_j$ for some finite subset J_0 of J, again by compactness of X. Then $\Box U_i$ and $(\Diamond V_j)_{j \in J_0}$ form the desired finite subcover.

In the former case, where $L_0 \notin \blacklozenge Q$, it must be that $L_0 \cap Q$ is empty, i.e., $Q \subseteq \bigcup_{j \in J} V_i$.

So $Q \subseteq \bigcup_{j \in J_0} V_i$ for some finite subset J_0 of J. Then $(\Diamond V_j)_{j \in J_0}$ is the desired finite subcover. □

When X is stably compact, observe that $\blacksquare Q$ is the complement of the subbasic open subset $\Diamond(X \setminus Q)$ of $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$, and that $\blacklozenge Q$ is the complement of the subbasic open subset $\Box(X \setminus Q)$ of $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$, so every closed subset of $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$ is cocompact in $\mathcal{P}\ell_{\mathcal{V}}(X)$: Lemma 4.3 entails immediately that the topology of $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$ is coarser than that of $\mathcal{P}\ell_{\mathcal{V}}(X)^{\mathsf{d}}$. Here is the converse statement.

Lemma 4.4. Let X be stably compact. Then every compact saturated subset $\mathcal{Q}$ of $\mathcal{P}\ell_{\mathcal{V}}(X)$ is closed in $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$.

Proof. Consider the map $i : \mathcal{P}\ell_{\mathcal{V}}(X) \to \mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ defined by $i(L) = (\uparrow L, cl(L))$. We claim that i is an embedding. First, the topology of $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ is generated by subbasic open sets of the form $\Box U \times \Diamond V$, where U and V are open in X, and $i^{-1}(\Box U \times \Diamond V) = \Box U \cap \Diamond V$, so i is continuous. The image of the subbasic open $\Box U$ of $\mathcal{P}\ell_{\mathcal{V}}(X)$ is $(\Box U \times \mathcal{H}_{\mathcal{V}}(X)) \cap \operatorname{Im} i$, while the image of $\Diamond V$ is $(\mathcal{Q}(X) \times \Diamond V) \cap \operatorname{Im} i$, which are open in $\operatorname{Im} i$, so i is an embedding.

Given any compact saturated subset $\mathcal{Q}$ of $\mathcal{P}\ell_{\mathcal{V}}(X)$, its image $i(\mathcal{Q})$ is compact, so its saturation $\uparrow i(\mathcal{Q})$ in $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ is compact saturated, hence closed in $(\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{d}} = \mathcal{Q}(X)^{\mathsf{d}} \times \mathcal{H}_{\mathcal{V}}(X)^{\mathsf{d}} = \mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}}) \times \mathcal{Q}(X^{\mathsf{d}})$, using Theorem 3.1. By the definition of the topology on $\mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}}) \times \mathcal{Q}(X^{\mathsf{d}})$, one can write the (open) complement $\mathcal{W}$ of $\uparrow i(\mathcal{Q})$ as a union of open rectangles $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are taken from bases of open sets of $\mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$, resp. $\mathcal{Q}(X^{\mathsf{d}})$. So $\mathcal{W}$ is a union of sets of the form $(\bigcap_{i=1}^{n} \Diamond V_i) \times \Box U$, where U, $V_1, \ldots, V_n$ are open in X^{d}.

The set $i^{-1}(\mathcal{W})$ is then a union of sets of the form $i^{-1}((\bigcap_{i=1}^{n} \Diamond V_i) \times \Box U) = \bigcap_{i=1}^{n} \Diamond V_i \cap \Box U$, where $U, V_1, \ldots, V_n$ are open in X^{d}. So $i^{-1}(\mathcal{W})$ is open in $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$. Its complement $i^{-1}(\uparrow i(\mathcal{Q}))$ is therefore closed in $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$.

It only remains to show that $i^{-1}(\uparrow i(\mathcal{Q})) = \mathcal{Q}$. The inclusion from right to left is obvious. Conversely, let $x \in i^{-1}(\uparrow i(\mathcal{Q}))$. Then there is an $y \in \mathcal{Q}$ such that $i(y) \leq i(x)$. As every topological embedding is also an order-embedding for the respective specialization orderings, $y \leq x$. Now $\mathcal{Q}$ is saturated, so $x \in \mathcal{Q}$. □

Putting Lemma 4.2, Lemma 4.3, and Lemma 4.4 together, we obtain:

Theorem 4.5 (Duality, Erratic Case). Let X be a stably compact space. Then $\mathcal{P}\ell_{\mathcal{V}}(X)^{\mathsf{d}} = \mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$.

This result can be analyzed more deeply as follows. Recall the embedding $i : \mathcal{P}\ell_{\mathcal{V}}(X) \to \mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$, mapping L to $(\uparrow L, cl(L))$, which we have used in the proof of Lemma 4.4. This allows us to consider $\mathcal{P}\ell_{\mathcal{V}}(X)$ as a subspace $\operatorname{Im} i$ of $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$, up to the homeomorphism i. The elements of $\operatorname{Im} i$ are certain pairs $(Q, F) \in \mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$, namely those such that $Q \cap F \neq \emptyset$, $Q \subseteq \uparrow(Q \cap F)$, and $F \subseteq cl(Q \cap F)$, and $\operatorname{Im} i$ is equipped with the induced topology.

Replacing X by X^{d}, there is also an embedding $j : \mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}}) \to \mathcal{Q}(X^{\mathsf{d}}) \times \mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$. By Lemma 4.2, $i(L) = (\uparrow L, \downarrow L)$. This allows us to give the following symmetrical description of j, remembering that the specialization ordering of X^{d} is the opposite of that of X: $j(L) = (\downarrow L, \uparrow L)$.

It follows that $\mathrm{Im}\, i \cong \mathcal{P}\ell_{\mathcal{V}}(X)$ and $\mathrm{Im}\, j \cong \mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$ are stably compact spaces, which are in one-to-one correspondence through the involution $\_^{\perp}$ that sends (Q, F) to (F, Q). In this sense, it is seen, more clearly than in Theorem 4.5, that again duality swaps angels and demons.

One finally notes that this idea yields an alternative proof to Theorem 4.5, which makes the role of $\_^{\perp}$ slightly clearer. We use here the notions of perfect and patch-continuous maps; we shall again use them later.

Definition 4.6 (Perfect map). Let X, Y be topological spaces. A map $f : X \to Y$ is *perfect* if and only if it is continuous, and the inverse image $f^{-1}(\mathcal{Q})$ of every compact saturated subset $\mathcal{Q}$ of Y is compact in X; f is *patch-continuous* whenever it is continuous from X^{patch} to Y^{patch}.

Every perfect map is patch-continuous, and order-preserving. In fact, when X and Y are locally compact, the perfect maps are exactly the patch-continuous, order-preserving maps, see (Jung, 2004, Proposition 2.14) or (Alvarez-Manilla et al., 2004, Proposition 13). A similar notion is that of a *proper* map (Gierz et al., 2003, Definition VI-6.20), which is a perfect map f such that, additionally, $\downarrow f(F)$ is closed for every closed subset F of X. When X is sober and Y is locally compact, the proper maps coincide with the perfect maps (Gierz et al., 2003, Lemma VI-6.21(ii)).

Alternate proof of Theorem 4.5. Consider $\mathrm{Im}\, i, \mathrm{Im}\, j$ with the induced topologies of their ambient spaces, respectively $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ and $\mathcal{Q}(X^{\mathsf{d}}) \times \mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$. We prove Theorem 4.5 under the following assumption, which we prove later:

($*$) Assumption: the topology of $(\mathrm{Im}\, i)^{\mathsf{d}}$ is induced from that of $(\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{d}}$.

Note that $(\mathrm{Im}\, i)^{\mathsf{d}}$ makes sense, as $\mathrm{Im}\, i$ is homeomorphic to the stably compact space $\mathcal{P}\ell_{\mathcal{V}}(X)$, and is therefore stably compact as well.

Let c be the swapping map, namely the homeomorphism that sends $(Q, F) \in \mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ to $(F, Q) \in \mathcal{H}_{\mathcal{V}}(X) \times \mathcal{Q}(X)$. Then j coincides with $c \circ i$.

Now, c is also a homeomorphism of the corresponding de Groot duals, i.e., of $(\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{d}}$ onto $(\mathcal{H}_{\mathcal{V}}(X) \times \mathcal{Q}(X))^{\mathsf{d}}$. By Theorem 3.1, c is also a homeomorphism from $\mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}}) \times \mathcal{Q}(X^{\mathsf{d}})$ onto $\mathcal{Q}(X^{\mathsf{d}}) \times \mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$. Its restriction to $\mathrm{Im}\, i$ is the homeomorphism $\_^{\perp}$ from $\mathrm{Im}\, i$ (with the induced topology of $\mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}}) \times \mathcal{Q}(X^{\mathsf{d}}) = (\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{d}}$) to $\mathrm{Im}\, j$ (with the induced topology of $\mathcal{Q}(X^{\mathsf{d}}) \times \mathcal{H}_{\mathcal{V}}(X^{\mathsf{d}})$). Using ($*$), $\_^{\perp}$ is therefore a homeomorphism from $(\mathrm{Im}\, i)^{\mathsf{d}}$ to $\mathrm{Im}\, j$.

Since i is a homeomorphism of $\mathcal{P}\ell_{\mathcal{V}}(X)$ onto $\mathrm{Im}\, i$, it is also one of $\mathcal{P}\ell_{\mathcal{V}}(X)^{\mathsf{d}}$ onto $(\mathrm{Im}\, i)^{\mathsf{d}}$, so $j^{-1} \circ \_^{\perp} \circ i$ is a homeomorphism of $\mathcal{P}\ell_{\mathcal{V}}(X)^{\mathsf{d}}$ onto $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$. However, $j^{-1} \circ \_^{\perp} \circ i$ is the identity map, and we conclude.

Although this proof is short, and makes the role of $\_^{\perp}$ clearer, it all depends on proving ($*$) above. Let us proceed.

We first claim that $i : \mathcal{P}\ell_{\mathcal{V}}(X) \to \mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ is perfect. Using similar arguments as in the proof of Lemma 4.4, the complement $\mathcal{W}$ of $\mathcal{Q}$ in $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ is a union of sets of the form $(\bigcap_{i=1}^n \Diamond V_i) \times \Box U$, where $U, V_1, \ldots, V_n$ are open in X^{d}. Then $i^{-1}(\mathcal{W})$ is a union of sets of the form $\bigcap_{i=1}^n \Diamond V_i \cap \Box U$, which are open in $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$. So its complement $i^{-1}(\mathcal{Q})$ is closed in $\mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$, and is therefore an intersection of finite unions of sets of the form $\blacksquare Q$ and $\blacklozenge Q$, Q compact saturated in X. By Lemma 4.3, the latter are compact saturated in $\mathcal{P}\ell_{\mathcal{V}}(X)$, and since $\mathcal{P}\ell_{\mathcal{V}}(X)$ is stably compact, $i^{-1}(\mathcal{Q})$ is also compact saturated.

It follows that i is patch-continuous. In particular, and since $\mathcal{P}\ell_{\mathcal{V}}(X)^{\mathsf{patch}}$ is compact, $\operatorname{Im} i$ is a compact subset of $(\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{patch}}$.

We now prove $(*)$, i.e., that the topology of $(\operatorname{Im} i)^{\mathsf{d}}$ is induced from that of $(\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{d}}$. It is enough to show that the compact saturated subsets $\mathcal{Q}$ of $\operatorname{Im} i$ are exactly the intersections of compact saturated subsets $\mathcal{Q}'$ of $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ with $\operatorname{Im} i$. In one direction, consider a compact saturated subset $\mathcal{Q}$ of $\operatorname{Im} i$, and take $\mathcal{Q}'$ equal to the saturation $\uparrow \mathcal{Q}$ of $\mathcal{Q}$ in $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$. Since $\mathcal{Q}$ is also compact in $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$, $\mathcal{Q}'$ is compact saturated, and it is easy to check that $\mathcal{Q} = \mathcal{Q}' \cap \operatorname{Im} i$. Conversely, we must show that every compact saturated subset $\mathcal{Q}'$ of $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ induces a compact saturated subset $\mathcal{Q} = \mathcal{Q}' \cap \operatorname{Im} i$ of $\operatorname{Im} i$. Note that $\mathcal{Q}'$ is closed in $(\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{patch}}$, and $\operatorname{Im} i$ is compact in the same space, so $\mathcal{Q} = \mathcal{Q}' \cap \operatorname{Im} i$ is compact in $(\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{patch}}$. However, the latter is T_2, so $\operatorname{Im} i$ is also closed in $(\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X))^{\mathsf{patch}}$. The subset $\mathcal{Q}$ of $\operatorname{Im} i$ is therefore compact also in $\operatorname{Im} i$. Since $\mathcal{Q}$ is clearly saturated in $\operatorname{Im} i$, we conclude: the compact saturated subsets of $\operatorname{Im} i$ are exactly the intersections of those of $\mathcal{Q}(X) \times \mathcal{H}_{\mathcal{V}}(X)$ with $\operatorname{Im} i$. $\square$

We again draw a domain-theoretic corollary in the realm of bicontinuous bicpos. Recall that $\mathcal{P}\ell(X)$ is the poset of all lenses ordered by the topological Egli-Milner ordering, and that strong lenses are those lenses L such that $\downarrow L = cl(L)$. On a coherent continuous dcpo X, $\mathcal{P}\ell(X)$ is a stably compact continuous dcpo, and the Vietoris and Scott topologies coincide. It follows immediately:

Corollary 4.7. Let X be a stably bicontinuous bicpo. Then $\mathcal{P}\ell(X)$ is a stably bicontinuous bicpo, and $\mathcal{P}\ell(X)^{\mathsf{op}} = \mathcal{P}\ell(X^{\mathsf{op}})$.

This time, and contrarily to the cases of the Smyth and Hoare powerdomains, we cannot conclude that $\mathcal{P}\ell(X)$ is a completely distributive lattice. In fact, $\mathcal{P}\ell(X)$ is not even bounded complete in general, see (Abramsky and Jung, 1994, Exercise 6.2.23(8)): take X to be the four-element lattice $\bot \leq a, b \leq \top$ with a and b incomparable, and realize that $\{(\bot, a), (\bot, b)\}$ and $\{(a, \bot), (b, \bot)\}$ are elements of $\mathcal{P}\ell(X)$ with two incomparable minimal upper bounds.

Again, the way-below and the converse of the way-above relations do not coincide. Take again $X = [0, 1]_\sigma$. The lens of X are subintervals $[a, b]$ of X with $a \leq b$, i.e., finitary lenses $\langle\{a, b\}\rangle$. Recall that, in general, the way-below relation on $\mathcal{P}\ell(X)$ specializes to finitary lenses by: $\langle E\rangle \ll \langle E'\rangle$ iff $E \subseteq \Downarrow E'$ and $E' \subseteq \Uparrow E$. So in $X = [0, 1]_\sigma$, $[a, b] \ll [a', b']$ iff $a \ll a'$ and $b \ll b'$. Using Corollary 4.7, $[a', b'] \gg [a, b]$ iff $a' \gg a$ and $b' \gg b$. So for example, $[0, 0] \ll [0, 0]$ but $[0, 0] \not\gg [0, 0]$, while $[1, 1] \gg [1, 1]$ but $[1, 1] \not\ll [1, 1]$.

5. Powerdomains for Erratic Choice II: A-Valuations

There is another way to prove the same results as above, using a functional view of the various powerdomains. This functional view was studied at length by Reinhold Heckmann in his PhD Thesis (Heckmann, 1990). The main reason why I am mentioning this is that, apart from the fact that this is a useful alternate viewpoint, this will give us the opportunity to introduce some technical tools that we shall need later on, but in a slightly simpler setting. However, I am aware that reading again about the Plotkin powerdomain is probably not an exciting prospect. In this case, the reader should proceed directly to Section 6, and return back to this section for missing information—in particular, to Proposition 5.5 on solutions of patch-continuous systems, Lemma 5.7 and Lemma 5.10 (Scott's formula), or Lemma 5.12 on retracts of stably compact spaces.

Let $\mathbb{S} = \{0, 1\}$ be Sierpiński space. The opens of $\mathbb{S}$ are $\emptyset$, $\mathbb{S}$ and $\{1\}$. Alternatively, $\mathbb{S}$ is the dcpo obtained from the ordering $0 < 1$; this is stably compact, and in fact another example of a completely distributive lattice, in particular of a stably bicontinuous bicpo. $\mathbb{S}^{\mathsf{patch}}$ has the discrete topology.

Consider the Smyth powerdomain $\mathcal{Q}(X)$. Every $Q \in \mathcal{Q}(X)$ defines a map $\mathfrak{u}_Q : \mathcal{O}(X) \to \mathbb{S}$ by $\mathfrak{u}_Q(U) = 1$ iff $Q \subseteq U$. (The notation is an anticipation on the *unanimity games* of Section 6. Heckmann's definition is the same, only replacing $\mathcal{O}(X)$ by the isomorphic space of all continuous maps from X to $\mathbb{S}$.) Then $\mathfrak{u}_Q$ is Scott-continuous, and preserves finite infs: $\mathfrak{u}_Q(U_1 \cap \ldots \cap U_n) = 1$ iff $\inf_{i=1}^n \mathfrak{u}_Q(U_i) = 1$. Conversely, every Scott-continuous, finite-inf-preserving map $\alpha : \mathcal{O}(X) \to \mathbb{S}$ is of the form $\mathfrak{u}_Q$ for some non-empty compact saturated set Q, as soon as X is sober. Consider indeed the Scott-open filter of all opens U such that $\alpha(U) = 1$, and use the Hofmann-Mislove theorem to show that their intersection is the desired element Q. It is also easy to show that this bijection between $\mathcal{Q}(X)$ and the space of Scott-continuous finite-inf-preserving maps is an order-isomorphism, and a homeomorphism once one equips the latter with the topology generated by $\Box U = \{\alpha \mid \alpha(U) = 1\}$, U open in X.

Similarly, there is an isomorphism between $\mathcal{H}(X)$ and the space of Scott-continuous, finite-sup-preserving maps (equivalently, the maps that preserve *all* sups): for every $F \in \mathcal{H}(X)$, let $\mathfrak{e}_F : \mathcal{O}(X) \to \mathbb{S}$ map every open U to 1 iff $F \cap U \neq \emptyset$. (We anticipate on the *example games* of Section 6.) One retrieves F such that $\mathfrak{e}_F = \alpha$ from any sup-preserving map α by letting F be the complement of the largest open set U such that $\alpha(U) = 0$. The topology on the space of sup-preserving maps that makes it homeomorphic to $\mathcal{H}_{\mathcal{V}}(X)$ is generated by $\Diamond U = \{\alpha \mid \alpha(U) = 1\}$, U open in X.

One can give a similar functional description of lenses L by a pair of maps $\mathfrak{u}_{\uparrow L}$, $\mathfrak{e}_{cl(L)}$. Alternatively, this is equivalent to a single map α from $\mathcal{O}(X)$ to $\mathbb{S} \times \mathbb{S}$, where $\alpha(U) = (1, 1)$ if $L \subseteq U$, $\alpha(U) = (0, 1)$ if $L \not\subseteq U$ but $L \cap U \neq \emptyset$, and $\alpha(U) = (0, 0)$ if $L \cap U = \emptyset$. The case $\alpha(U) = (0, 1)$, where $L \not\subseteq U$ and $L \cap U = \emptyset$, does not occur as lenses are non-empty.

So one can describe lenses by maps from $\mathcal{O}(X)$ to the subspace $\{(0, 0), (0, 1), (1, 1)\}$ of $\mathbb{S} \times \mathbb{S}$. Call this subspace $\mathbf{A}$, and rename $(0, 0)$ as 0, $(1, 1)$ as 1, and $(0, 1)$ as M: then we get exactly Heckmann's $\mathbf{A}$-valuations (Heckmann, 1997). The formal definition is as follows.

Definition 5.1 (A-valuation, $\mathcal{P}_{\mathcal{V}}(X)$). Let $\mathbf{A}$ be the dcpo with three elements 0, M, and 1, with ordering $\sqsubseteq$ such that $0 \sqsubseteq \mathsf{M} \sqsubseteq 1$. Let $\bigsqcup$, $\sqcup$ denote sup in $\mathbf{A}$.

An $\mathbf{A}$-*valuation* on the topological space X is a map α from $\mathcal{O}(X)$ to $\mathbf{A}$ such that:

1 α is *strict*: $\alpha(\emptyset) = 0$;
2 α is *normalized*: $\alpha(X) = 1$;
3 α is *monotone*: if $U \subseteq V$ then $\alpha(U) \sqsubseteq \alpha(V)$, for all opens U, V;
4 if U is an open such that $\alpha(U) = 0$ then $\alpha(U \cup V) = \alpha(V)$ for all opens V;
5 if U is an open such that $\alpha(U) = 1$ then $\alpha(U \cap V) = \alpha(V)$ for all opens V.

An $\mathbf{A}$-valuation α is *continuous* if and only if $\alpha(\bigcup_{i\in I} U_i) = \bigsqcup_{i \in I} \alpha(U_i)$ for every directed family of opens $(U_i)_{i\in I}$.

We let $\mathcal{P}(X)$ be the dcpo of all continuous $\mathbf{A}$-valuations, ordered pointwise, i.e., by $\sqsubseteq_{\mathbf{A}}$, defined by $\alpha \sqsubseteq_{\mathbf{A}} \alpha'$ iff $\alpha(U) \sqsubseteq \alpha'(U)$ for every open U of X.

We let $\mathcal{P}_{\mathcal{V}}(X)$ be the space of all continuous $\mathbf{A}$-valuations on X with the *Vietoris topology*, generated by:

$$\begin{aligned} \Box U &= \{\alpha \text{ continuous } \mathbf{A}\text{-valuation} \mid \alpha(U) = 1\} \\ \Diamond U &= \{\alpha \text{ continuous } \mathbf{A}\text{-valuation} \mid \alpha(U) \neq 0\} \end{aligned}$$

The Vietoris topology above is nothing else than the so-called topology of *pointwise convergence*. On any space F of functions from a space Z to a space Y, the topology of pointwise convergence is that induced by the product topology on Y^Z. This topology will play an important role in the rest of this paper. Note that if Y is compact, then Y^Z is compact by Tychonoff's Theorem; this will be the starting point of all our proofs of compactness. Note also that if F' is a subset of F, the induced topology is again the topology of pointwise convergence.

By definition, the topology of pointwise convergence on F has subbasic open sets of the form $[z \in V] = \{f \in F \mid f(z) \in V\}$, where $z \in Z$, and V is open in Y. When $Z = \mathcal{O}(X)$, $Y = \mathbf{A}$, and $F = \mathcal{P}_{\mathcal{V}}(X)$, the topology of pointwise convergence is therefore generated by the subbasic opens $[U \in \{\mathsf{M}, 1\}] = \Diamond U$ and $[U \in \{1\}] = \Box U$, $U \in \mathcal{O}(X)$. So, as claimed, this is the same as the Vietoris topology.

Let us briefly explain the connection between continuous $\mathbf{A}$-valuations and lenses. One may show that, when X is sober, $\mathcal{P}_{\mathcal{V}}(X)$ is naturally homeomorphic to the following space $\mathcal{P}'_{\mathcal{V}}(X)$. For lack of a better name, call *quasi-lens* on X any pair (Q, F) of a compact saturated subset Q of X and a closed subset F of X such that $L = Q \cap F$ is non-empty, $Q = \uparrow L$, and for every open U containing Q, $F \subseteq cl(U \cap F)$. Let $\mathcal{P}'_{\mathcal{V}}(X)$ be the space of quasi-lenses on X, with the topology generated by sets which we write again $\Box U$ and $\Diamond U$: $\Box U = \{(Q, F) \in \mathcal{P}'_{\mathcal{V}}(X) \mid Q \subseteq U\}$, $\Diamond U = \{(Q, F) \in \mathcal{P}'_{\mathcal{V}}(X) \mid F \cap U \neq \emptyset\}$. Accordingly, we call this topology the *Vietoris topology* on $\mathcal{P}'_{\mathcal{V}}(X)$.

We leave it as an exercise to the reader to show the following.

Fact 5.2. If X is sober, then $\mathcal{P}_{\mathcal{V}}(X)$ is homeomorphic to $\mathcal{P}'_{\mathcal{V}}(X)$. The homeomorphism is as follows. In one direction, every quasi-lens (Q, F) gives rise to a continuous $\mathbf{A}$-valuation $(Q, F)^*$, defined by $(Q, F)^*(U) = 1$ if $Q \subseteq U$, $(Q, F)^*(U) = 0$ if $F \cap U = \emptyset$, and $(Q, F)^*(U) = \mathsf{M}$ otherwise. Conversely, one retrieves a quasi-lens $(Q, F) = \alpha^\circ$ from any continuous **A**-valuation α by: Q is the intersection of all opens U such that $\alpha(U) = 1$, and F is the complement of the largest open U such that $\alpha(U) = 0$.

In a stably compact space, the notion of quasi-lens coincides with that of a lens, even with a strong lens, so there is not much point in dealing with quasi-lenses after all. I originally derived this from Theorem 5.21 below and Fact 5.2. The following direct proof is due to one of the anonymous referees.

Proposition 5.3. In a stably compact space X, every quasi-lens (Q, F) induces a lens $Q \cap F$; this is even a strong lens.

In particular, $\mathcal{P}\ell_{\mathcal{V}}(X)$ and $\mathcal{P}_{\mathcal{V}}(X)$ are homeomorphic: in one direction, for every lens L, let L^* be the continuous **A**-valuation defined by $L^*(U) = 1$ if $L \subseteq U$, $L^*(U) = 0$ if $L \cap U = \emptyset$, and $L^*(U) = \mathsf{M}$ otherwise; conversely, for every continuous **A**-valuation α, let α° be the lens $Q \cap F$, where Q is the intersection of all opens U such that $\alpha(U) = 1$, and F is the complement of the largest open U such that $\alpha(U) = 0$.

Proof. We first observe that: $(*)$ for every filtered family $(L_i)_{i \in I}$ of patch-compact subsets of X, $\downarrow \bigcap_{i \in I} L_i = \bigcap_{i \in I} \downarrow L_i$. The inclusion from left to right is obvious. Conversely, for any $x \in \bigcap_{i \in I} \downarrow L_i$, the family $(\uparrow x \cap L_i)_{i \in I}$ is again a filtered family of non-empty patch-compact subsets of X. Since X^{patch} is T_2, it is sober and therefore well-filtered, so $\bigcap_{i \in I} (\uparrow x \cap L_i)$ is a non-empty patch-compact subset of X. In particular, $\uparrow x \cap \bigcap_{i \in I} L_i$ is non-empty, meaning that $x \in \downarrow \bigcap_{i \in I} L_i$.

Let (Q, F) be a quasi-lens. By definition, $F \subseteq \bigcap_{\substack{U \in \mathcal{O}(X) \\ Q \subseteq U}} cl(U \cap F)$. Since X is locally compact, for every open subset U such that $Q \subseteq U$, there is a compact saturated subset Q_1 such that $Q \subseteq int(Q_1) \subseteq Q_1 \subseteq U$. So $F \subseteq \bigcap_{\substack{Q_1 \text{ compact saturated} \\ Q \subseteq int(Q_1)}} cl(Q_1 \cap F)$. Now for each compact saturated subset Q_1 whose interior contains Q, $Q_1 \cap F$ is patch-compact, in fact a lens. By Fact 4.1, $\downarrow(Q_1 \cap F)$ is closed, and therefore coincides with $cl(Q_1 \cap F)$. So $F \subseteq \bigcap_{\substack{Q_1 \text{ compact saturated} \\ Q \subseteq int(Q_1)}} \downarrow(Q_1 \cap F) = \downarrow \bigcap_{\substack{Q_1 \text{ compact saturated} \\ Q \subseteq int(Q_1)}} (Q_1 \cap F)$, using $(*)$. Again by local compactness, the latter is just $\downarrow(Q \cap F)$. Letting $L = Q \cap F$, we obtain $F \subseteq \downarrow L$, and therefore $F = \downarrow L$. Since $Q = \uparrow L$ by definition, L is a strong lens.

The rest of the Proposition is by composition with Fact 5.2. □

Heckmann managed to prove that his space of continuous **A**-valuations $\mathcal{P}_{\mathcal{V}}(X)$ is homeomorphic to the Plotkin powerdomain $\mathcal{P}\ell_{\mathcal{V}}(X)$ when X is a continuous dcpo (Heckmann, 1997, Corollary 6.2), and when X is Hausdorff (Heckmann, 1997, Theorem 5.1). Proposition 5.3 establishes this for stably compact spaces as well.

It follows from Theorem 4.5 that $\mathcal{P}_{\mathcal{V}}(X)$ too is self-dual. However, we now wish to prove this directly. This will be Theorem 5.21 below. As we have said above, this will give us the opportunity to introduce some tools we shall need in later sections. This will also make apparent the role of the involution $\_^\perp$.

We first need the following machinery of so-called patch-continuous systems of inequalities. This is a generalization of some techniques that Jung used to show that the probabilistic powerdomain of a stably compact space is stably compact (Jung, 2004;

Alvarez-Manilla et al., 2004), and which we shall use again and again. There is no doubt that one could generalize again, however we shall be content with the following.

Recall from Definition 4.6 that a map $f : Y \to Z$ is *patch-continuous* if and only if it is continuous from Y^{patch} to Z^{patch}.

Definition 5.4. Let T be a set, and A be a topological space. A *patch-continuous inequality* on T, A is any formula E of the form:

$$f(\_(t_1), \ldots, \_(t_m)) \dot{\leq} g(\_(t'_1), \ldots, \_(t'_n))$$

where f and g are patch-continuous maps from A^n to A, and $t_1, \ldots, t_m, t'_1, \ldots, t'_n$ are $m+n$ fixed elements of T. E *holds* at $\alpha : T \to A$ iff $f(\alpha(t_1), \ldots, \alpha(t_m)) \leq g(\alpha(t'_1), \ldots, \alpha(t'_n))$, where $\leq$ is the specialization quasi-ordering of A.

A *patch-continuous system* Σ on T, A is a (possibly infinite) set Σ of patch-continuous inequalities on T, A. Σ *holds* at $\alpha : T \to A$ iff every element of Σ holds at α.

We shall also consider *patch-continuous equations* of the form:

$$f(\_(t_1), \ldots, \_(t_m)) \doteq g(\_(t'_1), \ldots, \_(t'_n))$$

where f and g are as above. Any equation $a \doteq b$ can be rewritten as $a \dot{\leq} b$ and $b \dot{\leq} a$, so we shall freely use patch-continuous equations as well in defining patch-continuous systems.

Proposition 5.5. Let T be a set, A a stably compact space, and Σ a patch-continuous system on T, A. The subset $[\Sigma]$ of A^T of all maps $\alpha : T \to A$ such that Σ holds at α is patch-closed in A^T.

As such, $[\Sigma]$ is a stably compact subspace of A^T.

Proof. For every $t \in T$, the map $\alpha \in A^T \mapsto \alpha(t)$ is patch-continuous, as the continuous projection map from $(A^T)^{\mathsf{patch}} = (A^{\mathsf{patch}})^T$ to A^{patch}. For every $E \in \Sigma$, say $f(\_(t_1), \ldots, \_(t_m)) \dot{\leq} g(\_(t'_1), \ldots, \_(t'_n))$, the set $[E]$ of all α at which E holds is the inverse image by the patch-continuous map $\alpha \in A^T \mapsto (f(\alpha(t_1), \ldots, \alpha(t_m)), g(\_(t'_1), \ldots, \_(t'_n)))$ of the graph of $\leq$ in A^{patch}. Since A is stably compact, the latter is closed in A^{patch}, so $[E]$ is patch-closed in A^T. Since $[\Sigma] = \bigcap_{E \in \Sigma}[E]$, $[\Sigma]$ is also patch-closed in A^T.

It follows that $[\Sigma]$ is a stably compact subspace of A^T, since A^T is stably compact, and every patch-closed subspace of a stably compact space is stably compact (Jung, 2004, Proposition 2.16). □

It follows immediately:

Proposition 5.6. Let X be a topological space. The space $Aval(X)$ of all (not necessarily continuous) $\mathbf{A}$-valuations on X, with the induced topology from the product topology on $\mathbf{A}^{\mathcal{O}(X)}$, is stably compact.

Proof. First, $\mathbf{A}$ is stably compact, and its Nachbin pospace is $\{0, \mathsf{M}, 1\}$ with the discrete topology and $\sqsubseteq$ as ordering. So any map from $\mathbf{A}^n$ to $\mathbf{A}$ is patch-continuous. The space of $\mathbf{A}$-valuations on X is then $[\Sigma]$, where Σ consists of the following patch-continuous (in)equations with $T = \mathcal{O}(X)$, $A = \mathbf{A}$. First, $\_(\emptyset) \doteq 0$ (strictness), $\_(X) \doteq 1$ (normalization), $\_(U) \dot{\leq} \_(V)$ for every pair of opens U, V such that $U \subseteq V$ (monotonicity), $\_(U) \star \_(U \cup V) \doteq \_(U) \star \_(V)$ for all opens U, V of X, where $\star$ is the map (necessarily patch-continuous) from $\mathbf{A}^2$ to $\mathbf{A}$ defined by $0 \star y = y$, $\mathsf{M} \star y = 1 \star y = 1$ (property 4)—the $\star$ map was defined in (Heckmann, 1997, Theorem 3.2)—and finally $\_(U) \bar{\star} \_(U \cap V) \doteq \_(U) \bar{\star} \_(V)$ for all opens U, V of X, where $\bar{\star}$ is the map (necessarily patch-continuous) from $\mathbf{A}^2$ to $\mathbf{A}$ defined by $1 \bar{\star} y = y$, $\mathsf{M} \bar{\star} y = 0 \bar{\star} y = 0$ (property 5). Then apply Proposition 5.5. □

We now recall a popular form of *Scott's formula*.

Lemma 5.7 (Scott). Let Y be a poset in which every bounded directed family has a least upper bound, X a continuous poset, B a basis of X, and f a monotonic map from B to Y. Let:

$$\mathfrak{r}(f)(x) = \sup_{y \in B, y \ll x} f(y)$$

Then $\mathfrak{r}(f)$ is a Scott-continuous map from X to Y, and is the largest Scott-continuous map below f on B.

Let $\Subset$ be the way-below relation on $\mathcal{O}(X)$, for any given topological space X. The spaces X such that $\mathcal{O}(X)$ is a continuous dcpo are the *core-compact* spaces (Escardó and Heckmann, 2002, Section 5). Every locally compact space is core-compact. Moreover, if X is locally compact, then $U \Subset V$ iff $U \subseteq Q \subseteq V$ for some compact saturated subset Q (Gierz et al., 2003, Proposition I.1.4). We let the reader check that $V_1 \Subset U, \ldots, V_n \Subset U$ implies $\bigcup_{i=1}^n V_i \Subset U$, and that if X is core-compact (in particular locally compact), then $U \Subset \bigcup_{i=1}^n V_i$ iff there are opens $U_i \Subset V_i$, $1 \leq i \leq n$, such that $U \subseteq \bigcup_{i=1}^n U_i$.

Dual to core-compactness is a property that we would like to call core-coherence: A topological space X is *core-coherent* iff for all opens U, V_1, V_2 of X, if $U \Subset V_1$ and $U \Subset V_2$ then $U \Subset V_1 \cap V_2$. Any stably compact space is clearly both core-compact and core-coherent. This is a well-known property: this states that $\Subset$ is *multiplicative* on the lattice $\mathcal{O}(X)$, see (Abramsky and Jung, 1994, Definition 7.2.18) or (Gierz et al., 2003, Proposition I.4.7). Then we have the dual property that, if X is core-coherent, then $V \Subset \bigcap_{i=1}^n U_i$ ($n \geq 1$) iff there are opens $V_i \Subset U_i$, $1 \leq i \leq n$, such that $V \subseteq \bigcap_{i=1}^n V_i$.

The point is *preservation* of the way-below relation $\Subset$, and a *reflection* property, satisfied by various set operations.

Definition 5.8 (Preserving and Reflecting $\ll$). For any binary set operation $\circ$ on a poset Z, say that $\circ$ *preserves* $\ll$ iff $v_1 \ll u_1$ and $v_2 \ll u_2$ imply $v_1 \circ v_2 \ll u_1 \circ u_2$ for all v_1, u_1, v_2, u_2 in Z; $\circ$ *reflects* $\ll$ iff for any v, v_1, v_2 in Z such that $v \ll v_1 \circ v_2$, there are u_1 and u_2 in Z such that $v \leq u_1 \circ u_2$, $u_1 \ll v_1$, and $u_2 \ll v_2$.

The remarks above imply that, with $Z = \mathcal{O}(X)$, union $\cup$ preserves $\Subset$ in core-compact spaces, and intersection $\cap$ preserves $\Subset$ in core-coherent spaces. Union $\cup$ reflects $\Subset$ in core-compact spaces, and $\cap$ reflects $\Subset$ in core-coherent spaces. It is an easy exercise to show that whenever Z is a continuous dcpo, every Scott-continuous operation $\circ$ on Z reflects $\ll$.

Given that $\circ$ preserves and reflects $\Subset$, one sees that any expression of the form

$\sup_{V \Subset V_1 \circ V_2} f(V)$ can be rewritten as $\sup_{\substack{U_1 \Subset V_1 \\ U_2 \Subset V_2}} f(U_1 \circ U_2)$, for any monotonic map f from $\mathcal{O}(X)$ to some ordered space in which these sups make sense. We shall use this in the proof of Lemma 5.9 below, and in several other arguments. Indeed, since $\circ$ reflects $\Subset$, $\sup_{V \Subset V_1 \circ V_2} f(V) \leq \sup_{\substack{V, U_1, U_2 \\ V \subseteq U_1 \circ U_2 \\ V_1 \Subset U_1, V_2 \Subset U_2}} f(V) \leq \sup_{\substack{U_1, U_2 \\ V_1 \Subset U_1, V_2 \Subset U_2}} f(U_1 \circ U_2)$. And conversely, since $\circ$ preserves $\Subset$, $\sup_{\substack{U_1 \Subset V_1 \\ U_2 \Subset V_2}} f(U_1 \circ U_2) \leq \sup_{\substack{U_1, U_2 \\ U_1 \circ U_2 \Subset V_1 \circ V_2}} f(U_1 \circ U_2) \leq \sup_{V \Subset V_1 \circ V_2} f(V)$.

Lemma 5.9. Let X be a stably compact space. For every $\mathbf{A}$-valuation α on X, $\mathfrak{r}(\alpha)$ is a continuous $\mathbf{A}$-valuation on X.

Proof. By Lemma 5.7, $\mathfrak{r}(\alpha)$ is monotone and continuous. By definition, $\mathfrak{r}(\alpha)(U) = \sup_{V \Subset U} \alpha(V)$. That $\mathfrak{r}(\alpha)$ is strict is clear. Since X is compact, $X \Subset X$, so $\mathfrak{r}(\alpha)(X) = 1$, hence $\mathfrak{r}(\alpha)$ is normalized. If $\mathfrak{r}(\alpha)(U) = 0$, then $\alpha(W_1) = 0$ for every $W_1 \Subset U$, so:

$$
\begin{aligned}
\mathfrak{r}(\alpha)(U \cup V) &= \sup_{W \Subset U \cup V} \alpha(W) \\
&= \sup_{W_1 \Subset U, W_2 \Subset V, W \subseteq W_1 \cup W_2} \alpha(W) \\
&\quad \text{since } \cup \text{ preserves and reflects } \Subset \text{ (core-compactness)} \\
&= \sup_{W_1 \Subset U, W_2 \Subset V} \alpha(W_1 \cup W_2) \quad \text{since } \alpha \text{ is monotone} \\
&= \sup_{W_1 \Subset U, W_2 \Subset V} \alpha(W_2) \quad \text{since } \alpha(W_1) = 0 \text{ for every } W_1 \Subset U, \\
&= \sup_{W_2 \Subset V} \alpha(W_2) = \mathfrak{r}(\alpha)(V)
\end{aligned}
$$

Similarly, if $\mathfrak{r}(\alpha)(U) = 1$, then $\mathfrak{r}(\alpha)(U \cap V) = \mathfrak{r}(V)$, using core-coherence instead of core-compactness, i.e., the fact that $\cap$ preserves and reflects $\Subset$. □

In general, any property P on maps $f : \mathcal{O}(X) \to \mathbf{A}$ that can be expressed by only using operations that preserve and reflect $\Subset$ in $\mathcal{O}(X)$, application of f to opens of X, and Scott-continuous maps in $\mathbf{A}$, also holds of $\mathfrak{r}(f)$. Lemma 5.9 is the particular case where P is the property of being an $\mathbf{A}$-valuation.

A *retract* of a topological space Y is a topological space Z such that there are two continuous maps $s : Z \to Y$ (the *section*) and $r : Y \to Z$ (the *retraction*) such that $r(s(z)) = z$ for all $z \in Z$.

Scott's formula yields a retraction almost for free. Recall that the topology of pointwise convergence on any space Z of functions from X to Y is induced from the product topology, and has subbasic open sets $[x \in V] = \{f \in Z \mid f(x) \in V\}$, $x \in X$, V open in Y. We write $[x \in V]_Z$ in case the ambient space Z is ambiguous.

Lemma 5.10. Let Y be a poset in which every bounded directed family has a least upper bound, X a continuous poset, B a basis of X, and define $\mathfrak{r}$ as in Lemma 5.7.

For every set Z of monotonic maps from X to Y such that $\mathfrak{r}(Z) \subseteq Z$, $\mathfrak{r}$ defines a retraction from Z onto its image $\mathfrak{r}(Z)$, where Z and $\mathfrak{r}(Z)$ are equipped with the topology of pointwise convergence. The inclusion $\mathfrak{s} : \mathfrak{r}(Z) \subseteq Z$ is the associated section.

Proof. First, $\mathfrak{r}$ is continuous: for any subbasic open set $[x \in V]_{\mathfrak{r}(Z)}$, its inverse image by $\mathfrak{r}$ is $\{f \in Z \mid \mathfrak{r}(f)(x) \in V\} = \{f \in Z \mid \exists y \in B, y \ll x, f(y) \in V\} = \bigcup_{y \in B, y \ll x} [y \in f^{-1}(V)]_Z$. Second, $\mathfrak{s}$ is continuous, because the topology of pointwise convergence on $\mathfrak{r}(Z) \subseteq Z$ is induced by that on Z. Finally, we claim that $\mathfrak{r}(\mathfrak{s}(f)) = f$ for all $f \in \mathfrak{r}(Z)$. We must show that $\mathfrak{r}(\mathfrak{s}(f))(x) = f(x)$ for all $x \in X$. Since $f \in \mathfrak{r}(Z)$, f is Scott-continuous. Then $\mathfrak{r}(\mathfrak{s}(f))(x) = \sup_{y \in B, y \ll x} f(y) = f(\sup_{y \in B, y \ll x} y) = f(x)$. □

Lemma 5.11. Let X be stably compact. The map $\mathfrak{r}$ is continuous from $Aval(X)$ to $\mathcal{P}_{\mathcal{V}}(X)$. It is a retraction, with the canonical inclusion $\mathfrak{s}$ as associated section.

Proof. Recall that $Aval(X)$ is the space of all **A**-valuations on X, and that the Vietoris topology is the topology of pointwise convergence. Apply Lemma 5.10 to $Z = Aval(X)$, where $\mathfrak{r}(Z) \subseteq Z$ follows from Lemma 5.9. It remains to show that $\mathfrak{r}(Z)$ coincides with $\mathcal{P}_{\mathcal{V}}(X)$ as a set. It is enough to observe that every continuous **A**-valuation α coincides with $\mathfrak{r}(\alpha)$, hence is in $\mathfrak{r}(Z)$. □

We now use the following result, which we shall call *Lawson's Lemma* (Lawson, 1987, Proposition, bottom of p.153, and subsequent discussion), see also (Jung, 2004, Proposition 2.17).

Lemma 5.12 (Lawson). Any topological space that arises as a retract of a stably compact space is itself stably compact.

In fact, taking retracts preserves any property among sobriety, local compactness, coherence, compactness.

By Proposition 5.6, the space of all **A**-valuations is stably compact. By Lemma 5.11, $\mathcal{P}_{\mathcal{V}}(X)$ is a retract of it, as soon as X is stably compact. Using Lawson's Lemma 5.12, it follows immediately:

Proposition 5.13. If X is stably compact, then so are $\mathcal{P}_{\mathcal{V}}(X)$ and $\mathcal{P}'_{\mathcal{V}}(X)$.

Notice that this was proved exactly as Jung proved that the probabilistic powerdomain of a stably compact space is stably compact (Jung, 2004; Alvarez-Manilla et al., 2004). Recall that Mislove proved that $\mathcal{P}\ell(X)$ is stably compact (Mislove, 1998, Corollary 4.48), however this requires not just X to be stably compact, but also to be a continuous dcpo. We dispensed with the latter assumption in Proposition 5.13. One can reprove Mislove's theorem by noting that, when X is a continuous dcpo, $\mathcal{P}_{\mathcal{V}}(X)$ is isomorphic to $\mathcal{P}\ell(X)$ (Heckmann, 1997, Corollary 6.2).

We now come to actual duality on $\mathcal{P}_{\mathcal{V}}(X)$. Write $\bigsqcap$, $\sqcap$ for infs in **A**.

Definition 5.14. Let X be a topological space. Given any continuous **A**-valuation α, let $\alpha^\dagger(Q)$ be defined, for every compact saturated subset Q of X, by:

$$\alpha^\dagger(Q) = \bigsqcap_{\substack{U \in \mathcal{O}(X) \\ Q \subseteq U}} \alpha(U).$$

This definition is inspired from a similar definition for games, see Section 6; in the case of valuations, this was introduced in (Tix, 1995).

Lemma 5.15. Let X be stably compact. For every compact saturated subset Q of X, $\blacksquare Q = \{\alpha \in \mathcal{P}_{\mathcal{V}}(X) \mid \alpha^\dagger(Q) = 1\}$ and $\blacklozenge Q = \{\alpha \in \mathcal{P}_{\mathcal{V}}(X) \mid \alpha^\dagger(Q) \neq 0\}$ are compact saturated in $\mathcal{P}_{\mathcal{V}}(X)$.

Proof. For every $a \in \mathbf{A}$, consider the set of all (not necessarily continuous) $\mathbf{A}$-valuations α such that $a \sqsubseteq \alpha(U)$ for all opens U of X containing Q. This is the subset $[\Sigma_{Q \sqsupseteq a}]$, where $\Sigma_{Q \sqsupseteq a}$ is the system of patch-continuous equations obtained by adding the patch-continuous inequalities $a \dot{\leq} \_(U)$, for each open U of X containing Q, to the system Σ used in the proof of Proposition 5.6. By Proposition 5.5, $[\Sigma_{Q \sqsupseteq a}]$ is a stably compact space with the topology induced by the product topology on $\mathbf{A}^{\mathcal{O}(X)}$.

Now let $\langle Q \sqsupseteq a\rangle$ denote the space of all *continuous* $\mathbf{A}$-valuations α on X such that $a \sqsubseteq \alpha(U)$ for all opens U of X containing Q. We equip it with the induced topology from $\mathcal{P}_{\mathcal{V}}(X)$; note that this is another Vietoris topology, generated from subsets that one may again write $\Box U$ and $\Diamond U$ for each open U of X, respectively defined as $\{\alpha \in \langle Q \sqsupseteq a\rangle \mid \alpha(U) = 1\}$ and as $\{\alpha \in \langle Q \sqsupseteq a\rangle \mid \alpha(U) \neq 0\}$. By Lemma 5.10, and following the same argument as in Lemma 5.11, $\mathfrak{r}$ is a retraction of $[\Sigma_{Q \sqsupseteq a}]$ onto $\langle Q \sqsupseteq a\rangle$, with the canonical inclusion $\mathfrak{s}$ as associated section. We have to check that $\mathfrak{r}$ maps each element α of $[\Sigma_{Q \sqsupseteq a}]$ to one in $\langle Q \sqsupseteq a\rangle$. Indeed, let U be an arbitrary open containing Q. As in any locally compact space, there is a compact saturated subset Q' of X such that $Q \subseteq int(Q') \subseteq Q' \subseteq U$. Then $\mathfrak{r}(\alpha)(U) = \sup_{V \Subset U} \alpha(V)$ is greater than or equal to $\alpha(int(Q'))$, which is greater than or equal to a, by assumption.

By Lawson's Lemma 5.12, $\langle Q \sqsupseteq a\rangle$ is a stably compact space. It follows that, qua subset of $\mathcal{P}_{\mathcal{V}}(X)$, $\langle Q \sqsupseteq a\rangle$ is compact. It is also clearly saturated. We conclude that $\blacksquare Q$, which equals $\langle Q \sqsupseteq 1\rangle$, and $\blacklozenge Q$, which equals $\langle Q \sqsupseteq \mathsf{M}\rangle$, are compact saturated. $\Box$

We shall use similar compactness arguments, using retracts of spaces defined by systems of patch-continuous inequalities, for valuations, games and previsions.

In Proposition 5.18 below, we show a converse to Lemma 5.15: every compact saturated subset of $\mathcal{P}_{\mathcal{V}}(X)$ can be obtained as an intersection of finite unions of subsets of the form $\blacksquare Q$ or $\blacklozenge Q$. We require yet another pair of lemmas. Let $[Z \to Y]$ denote the space of continuous maps from Z to the dcpo Y, with the Scott topology of the pointwise ordering. Let $[Z \to Y]_{\mathsf{p}}$ be the same space, but with the topology induced from the product topology on Y^Z. (The subscript p is for "product", or for "pointwise convergence".)

Lemma 5.16. Let A be a bc-domain with bottom element $\perp$, and Z be a continuous poset, e.g., $Z = \mathcal{O}(X)$ for some locally compact space X. The Scott topology coincides with the product topology, i.e., $[Z \to A] = [Z \to A]_{\mathsf{p}}$, and $[Z \to A]$ is a bc-domain.

Proof. First, every subbasic open $\{f \in [Z \to A] \mid f(z) \in V\}$ of the product topology ($z \in Z, V$ open in A) is clearly Scott-open. So the Scott topology is finer than the product topology.

Conversely, call a *step* any map of the form $a \searrow b$, where $a \searrow b$ maps each $z \in Z$ to b if $a \ll z$, to $\perp$ otherwise. Call *step function* any map that is the (pointwise) least upper bound of finitely many steps. Under the conditions of the Lemma, $[Z \to A]$ is a bc-domain with a basis of step functions; this is almost standard: the case where Z is a continuous dcpo is dealt with in (Gierz et al., 2003, Exercise II-2.31), but continuous posets are dealt with in exactly the same way. In fact, one shows that every element f of $[Z \to A]$ is the least upper bound of a directed family of step functions $\sup_{i=1}^{n}(x_i \searrow y_i)$ satisfying the stronger condition that $y_i \ll f(x_i)$, $1 \leq i \leq n$. We claim that this entails that every Scott-open U is open in the product topology. Indeed, for every $f \in U$, one can find a step function $g^f = \sup_{i=1}^{n^f}(x_i^f \searrow y_i^f)$ such that $y_i^f \ll f(x_i^f)$ for every i, $1 \leq i \leq n^f$, and such that g^f is in U. We show that $U = \bigcup_{f \in U} \bigcap_{i=1}^{n_f} [x_i^f \mapsto \Uparrow y_i^f]$, where $[x \mapsto V]$ denotes the subbasic open of the product topology, consisting of all maps sending x to an element of the open V.

For every $f \in U$, $f \in \bigcap_{i=1}^{n_f} [x_i^f \mapsto \Uparrow y_i^f]$, since $y_i^f \ll f(x_i^f)$ for every i, $1 \leq i \leq n^f$. Conversely, for every g that is in $\bigcap_{i=1}^{n_f} [x_i^f \mapsto \Uparrow y_i^f]$ for some $f \in U$, the fact that $y_i^f \ll g(x_i^f)$ for each i, $1 \leq i \leq n^f$, entails $g^f \ll g$ (Gierz et al., 2003, Exercise II-2.31 (ii)). Since $g^f \in U$, $g \in U$, and we conclude. That $[Z \to A]$ is a bc-domain is clear: it is a continuous dcpo with basis given by step functions, and it is easy to see that it is bounded-complete, where existing sups are computed pointwise. □

Lemma 5.17. Let A be a continuous complete lattice, and X a locally compact space. Let $\langle Q \geq a \rangle^*$ be the set of elements α of $[\mathcal{O}(X) \to A]$ such that $a \sqsubseteq \alpha(U)$ for all opens U containing Q. Then every compact saturated subset of $[\mathcal{O}(X) \to A]$ is an intersection of finite unions of sets of the form $\langle Q \geq a \rangle^*$, Q compact saturated in X, $a \in A$.

Proof. Let α be any element of $[\mathcal{O}(X) \to A]$. For each fixed compact saturated subset Q of X, define $\alpha^\dagger$ as the inf of all $\alpha(U)$, where U ranges over the opens containing Q. This extends the above definition of $\alpha^\dagger$.

Note that $\alpha \leq \beta$ iff $\alpha^\dagger \leq \beta^\dagger$. The only if direction is clear. In the if direction, for every open U, U is the sup of the directed family of all opens $V \Subset U$. Since X is locally compact, if $V \Subset U$ then there is a compact saturated subset Q such that $V \subseteq Q \subseteq U$. Then, $\alpha(V) \leq \alpha^\dagger(Q)$ by the definition of $\alpha^\dagger$, that $\alpha^\dagger(Q) \leq \beta^\dagger(Q)$ by assumption, and $\beta^\dagger(Q) \leq \beta(U)$ by definition. Since $\alpha(U) = \sup_{V \Subset U} \alpha(V)$, $\alpha(U) \leq \beta(U)$.

It follows that $\uparrow \alpha = \bigcap_Q \langle Q \geq \alpha^\dagger(Q) \rangle^*$, where Q ranges over all compact saturated subsets of X. Then, for every finite subset $\mathcal{E} = \{\alpha_1, \ldots, \alpha_n\}$ of $[\mathcal{O}(X) \to A]$, $\uparrow \mathcal{E} = \bigcup_{i=1}^{n} \bigcap_Q \langle Q \geq \alpha_i^\dagger(Q) \rangle^* = \bigcap_{Q_1, \ldots, Q_n} (\langle Q_1 \geq \alpha_1^\dagger(Q_1) \rangle^* \cup \ldots \cup \langle Q_n \geq \alpha_n^\dagger(Q_n) \rangle^*)$. Since X is locally compact, $Z = \mathcal{O}(X)$ is a continuous dcpo, so $[\mathcal{O}(X) \to A]$ is a continuous bc-domain. In particular, any compact saturated subset $\mathcal{Q}$ of $[\mathcal{O}(X) \to A]$ is a filtered intersection of subsets of the form $\uparrow \mathcal{E}$, $\mathcal{E}$ finite, hence is an intersection of finite intersections of subsets of the form $\langle Q \geq a \rangle^*$. □

Proposition 5.18. Let X be stably compact. The compact saturated subsets of $\mathcal{P}_\mathcal{V}(X)$ are exactly the intersections of finite unions of sets of the form $\blacksquare Q$ or $\blacklozenge Q$, Q compact saturated in X. In other words, the topology of $\mathcal{P}_\mathcal{V}(X)^\mathsf{d}$ is generated by complements of sets of the form $\blacksquare Q$ or $\blacklozenge Q$, Q compact saturated in X.

Proof. One direction is Lemma 5.15. Conversely, let $\mathcal{Q}$ be a compact saturated subset of $\mathcal{P}_\mathcal{V}(X)$. As a subset of $[\mathcal{O}(X) \to \mathbf{A}]_\mathsf{p}$, $\mathcal{Q}$ is again compact, since the topology of $\mathcal{P}_\mathcal{V}(X)$ is induced from the product topology. Write $\uparrow \mathcal{Q}$ the upward-closure of $\mathcal{Q}$ in $[\mathcal{O}(X) \to \mathbf{A}]_\mathsf{p}$. This is compact saturated in $[\mathcal{O}(X) \to \mathbf{A}]_\mathsf{p}$. Now $\mathbf{A}$ is a bc-domain with a least element, and X is locally compact, so $[\mathcal{O}(X) \to \mathbf{A}]_\mathsf{p} = [\mathcal{O}(X) \to \mathbf{A}]$ by

Lemma 5.16. So $\uparrow \mathcal{Q}$ is compact saturated in $[\mathcal{O}(X) \to \mathbf{A}]$. By Lemma 5.17, $\uparrow \mathcal{Q}$ is an intersection of finite unions of sets of the form $\langle Q \geq a \rangle^*$, Q compact saturated in X, $a \in \mathbf{A}$. Heckmann observed that the specialization ordering of $\mathcal{P}_{\mathcal{V}}(X)$ is the pointwise ordering $\sqsubseteq_{\mathbf{A}}$ (Heckmann, 1997, Section 3.3). This is the same specialization ordering as in $[\mathcal{O}(X) \to \mathbf{A}]$. Since $\mathcal{Q}$ is saturated in $\mathcal{P}_{\mathcal{V}}(X)$, $\mathcal{Q}$ is the intersection of $\uparrow \mathcal{Q}$ with $\mathcal{P}_{\mathcal{V}}(X)$. So $\mathcal{Q}$ is an intersection of finite unions of sets of the form $\langle Q \geq a \rangle^* \cap \mathcal{P}_{\mathcal{V}}(X)$. We conclude since the latter equals $\blacksquare Q$ when $a = 1$, $\blacklozenge Q$ when $a = \mathsf{M}$, and the whole of $\mathcal{P}_{\mathcal{V}}(X)$ (i.e., $\blacksquare X$) when $a = 0$. $\square$

The following is directly defined from a definition we shall see later on games (Definition 6.9), much as the definition of $\alpha^\dagger$ was already (Definition 5.14).

Definition 5.19. Let X be a stably compact space. For every continuous $\mathbf{A}$-valuation α on X, let $\alpha^\perp$ be the map from $\mathcal{O}(X^{\mathsf{d}})$ to $\mathbf{A}$ defined by:

$$\alpha^\perp(X \setminus Q) = 1 - \alpha^\dagger(Q),$$

where the map $1 - \_$ on $\mathbf{A}$ is defined by $1 - 1 = 0$, $1 - 0 = 1$, $1 - \mathsf{M} = \mathsf{M}$.

Lemma 5.20. Let X be stably compact. For every continuous $\mathbf{A}$-valuation α on X, $\alpha^\perp$ is a continuous $\mathbf{A}$-valuation on X^{d}, and $\alpha^{\perp\perp} = \alpha$.

Proof. Strictness: $\alpha^\perp(\emptyset) = 1 - \alpha^\dagger(X) = 1 - 1 = 0$. Normalization: $\alpha^\perp(X^{\mathsf{d}}) = 1 - \alpha^\dagger(\emptyset) = 1 - 0 = 1$. Monotonicity: if $X \setminus Q \subseteq X \setminus Q'$, then $Q' \subseteq Q$, so that $\alpha^\dagger(Q') = \bigsqcap_{U \supseteq Q'} \alpha(U) \sqsubseteq \bigsqcap_{U \supseteq Q} \alpha(U) = \alpha^\dagger(Q)$, hence $\alpha^\perp(X \setminus Q) \sqsubseteq \alpha^\perp(X \setminus Q')$. (Subscripts such as $U \supseteq Q'$ abbreviate an enumeration of all open subsets U of X such that U contains Q'.) Continuity: let $(X \setminus Q_i)_{i \in I}$ be a directed family of opens in X^{d}, i.e., $(Q_i)_{i \in I}$ is a filtered family of compact saturated subsets of X. By well-filteredness, for any open U of X, $\bigcap_{i \in I} Q_i \subseteq U$ iff $Q_i \subseteq U$ for some $i \in I$. So $\alpha^\dagger(\bigcap_{i \in I} Q_i) = \bigsqcap_{U \supseteq \bigcap_{i \in I} Q_i} \alpha(U) = \bigsqcap_{U \text{ such that } Q_i \subseteq U \text{ for some } i \in I} \alpha(U) = \bigsqcap_{i \in I} \bigsqcap_{U \supseteq Q_i} \alpha(U) = \bigsqcap_{i \in I} \alpha^\dagger(Q_i)$. It follows that $\alpha^\perp(\bigcup_{i \in I}(X \setminus Q_i)) = 1 - \bigsqcap_{i \in I} \alpha^\dagger(Q_i) = \bigsqcup_{i \in I} \alpha^\perp(X \setminus Q_i)$.

Property 4: if $\alpha^\perp(X \setminus Q) = 0$, then $\alpha(V) = 1$ for all opens V that contain Q. For any open U containing $Q \cap Q'$, there are two opens V, V' such that $Q \subseteq V$, $Q' \subseteq V'$, and $V \cap V' \subseteq U$. This property is well-known to hold on stably compact spaces, see e.g. (Keimel and Lawson, 2005, Lemma 8.1), where spaces satisfying this are called *weakly Hausdorff*. Then, for any compact saturated subset Q', $\alpha^\dagger(Q \cap Q') = \bigsqcap_{U \supseteq Q \cap Q'} \alpha(U) = \bigsqcap_{V \supseteq Q, V' \supseteq Q', U \supseteq V \cap V'} \alpha(U)$. Since $\alpha(V) = 1$ for every $V \supseteq Q$, $\alpha(V \cap V') = \alpha(V')$, so $\alpha^\dagger(Q \cap Q')$ is larger than or equal to $\bigsqcap_{V' \supseteq Q'} \alpha(V') = \alpha^\dagger(Q')$. It follows that $\alpha^\perp((X \setminus Q) \cup (X \setminus Q')) = \alpha^\perp(X \setminus Q')$.

Property 5 is proved similarly, using the fact that for any open U containing $Q \cup Q'$, there are two opens V, V' such that $Q \subseteq V$, $Q' \subseteq V'$, and $V \cup V' \subseteq U$. (Take $V = V' = U$.)

Finally, we show that $\alpha^{\perp\perp} = \alpha$. Note that $\alpha^\perp$ is a continuous $\mathbf{A}$-valuation on X^{d}, and that for every continuous $\mathbf{A}$-valuation β on X^{d}, $\beta^\perp(U)$ is defined as $1 - \bigsqcap_{X \setminus Q \supseteq X \setminus U} \beta(X \setminus Q)$, i.e., $\beta^\perp(U) = \bigsqcup_{Q \subseteq U}(1 - \beta(X \setminus Q))$. So, for every open U of X, $\alpha^{\perp\perp}(U)$ equals $\bigsqcup_{Q \subseteq U} \bigsqcap_{V \supseteq Q} \alpha(V)$. In particular, the inequality $\alpha^{\perp\perp}(U) \sqsubseteq \alpha(U)$ is clear (take $V = U$ in the inf). Conversely, since α is continuous, $\alpha(U) = \bigsqcup_{W \Subset U} \alpha(W)$. For each open $W \Subset U$, there is a compact saturated Q such that $W \subseteq Q \subseteq U$. Then $\alpha(W) \sqsubseteq \bigsqcap_{V \supseteq Q} \alpha(V)$, so $\alpha(W) \sqsubseteq \alpha^{\perp\perp}(U)$, and we conclude. ☐

We call $\alpha^\perp$ the *dual* of α. This is the third time we see de Groot duality acting on models of choice, and perhaps the first non-trivial instance.

For short, say that $\_^\perp$ is *involutive* whenever $\_^{\perp\perp}$ is the identity map.

Theorem 5.21 (Duality, Erratic Case). Let X be a stably compact space. Then $\_^\perp$ defines an involutive homeomorphism from $\mathcal{P}_\mathcal{V}(X)^\mathsf{d}$ to $\mathcal{P}_\mathcal{V}(X^\mathsf{d})$.

Proof. The inverse image by $\_^\perp$ of the subbasic open $\Box(X \setminus Q)$ (Q compact saturated in X) is the complement of $\blacklozenge Q$, and conversely the image of $\blacklozenge Q$ by $\_^\perp$ is $\Box(X \setminus Q)$. Similarly, the inverse image of $\Diamond(X \setminus Q)$ is the complement of $\blacksquare Q$, and the image of $\blacksquare Q$ is $\Diamond(X \setminus Q)$. By Proposition 5.18, $\_^\perp$ is therefore continuous and open. It is also involutive by Lemma 5.20, hence a homeomorphism. ☐

So the $\mathcal{P}_\mathcal{V}$ construction is self-dual. Because of the homeomorphism of Fact 5.2, all this transports to spaces of quasi-lenses. We make this explicit: duality $\_^\perp$ operates on quasi-lenses by exchanging demons (the Q part) and angels (the F part).

Proposition 5.22 (Exchanging Angels and Demons). Let X be a stably compact space. For any quasi-lens (Q, F) on X, let its *dual* $(Q, F)^\perp$ be (F, Q). This is a quasi-lens on X^d, we have $(Q, F)^{\perp\perp} = (Q, F)$, and $(Q, F)^\perp = (Q, F)^{*\perp\circ}$, where the $\_^\perp$ operation on the right-hand side is the one of Lemma 5.20, and $\_^*$ and $\_^\circ$ are defined in Fact 5.2.

Proof. We do this in reverse, and check that $(Q, F)^{*\perp\circ}$ is indeed equal to (F, Q). From this all other assertions follow immediately. Let $\alpha = (Q, F)^*$. We first compute $\alpha^\dagger$. For every compact saturated subset Q_1 of X, $\alpha^\dagger(Q_1) = 1$ if and only if for every open U containing Q_1, $\alpha(U) = 1$, iff every open U that contains Q_1 also contains Q. Since Q_1 is saturated, this is equivalent to $Q \subseteq Q_1$. Next, $\alpha^\dagger(Q_1) = 0$ if and only if some open U that contains Q_1 fails to intersect F, iff the complement V of F contains Q_1, iff $Q_1 \cap F = \emptyset$. So $\alpha^\dagger$ is defined just as α is: it maps Q_1 to 1 if $Q \subseteq Q_1$, to 0 if $Q_1 \cap F = \emptyset$, and to M otherwise.

For clarity, we now use primes in denoting subsets of X^d. E.g., while the opens of X^d are just cocompacts of the form $X \setminus Q_1$, Q_1 compact saturated in X, denote them by symbols such as U', V'.

The above computation of $\alpha^\dagger$ entails that $\beta = \alpha^\perp$ maps each open U' of X^d to 1 if $F \subseteq U'$, to 0 if $U' \cap Q = \emptyset$, and to M otherwise. Let $(Q', F') = \beta^\circ$. Note that F is compact saturated in X^d. Since F is saturated, it is the intersection of all opens U' that contain it, whence $Q' = F$. Note also that Q is closed in X^d, hence its complement V' is open in X^d. The union of all opens U' of X^d such that $U' \cap Q = \emptyset$ is just V'. But then F' is the complement of V' by definition of $\_^\circ$, so $Q = F'$. ☐

Fact 5.2 then entails the following consequence of Theorem 5.21:

Corollary 5.23. Let X be a stably compact space. Then $\_^\perp$ defines an involutive homeomorphism from $\mathcal{P}'_\mathcal{V}(X)^\mathsf{d}$ to $\mathcal{P}'_\mathcal{V}(X^\mathsf{d})$.

We draw the attention of the reader to the fact that the definition of duality on spaces of quasi-lenses is far from trivial. Don't be fooled by the apparent simplicity of the formula $(Q, F)^\perp = (F, Q)$. If we had to check directly that (F, Q) is indeed a quasi-lens on X^{d}, the second condition to check would be that F is the upward-closure in the specialization ordering of X^{d}, namely the downward-closure of $L = F \cap Q$. The only thing we know is that (third condition), for any open U containing Q, F is contained in $cl(U \cap F)$. Stable compactness is needed to go from the weak latter condition to the strong, former condition.

Now, a consequence of this is any quasi-lens (Q, F) is such that $L = Q \cap F$ is non-empty, $Q = \uparrow L$, and $F = \downarrow L$. Since F is closed, and $\downarrow L \subseteq cl(L)$, we also obtain $F = cl(L)$. As we have already said, such quasi-lenses are in one-to-one correspondence with lenses: map (Q, F) to L in one direction, L to $(\uparrow L, cl(L))$ in the other direction. Subbasic open sets $\Box U$, $\Diamond U$ are again preserved in every direction. Then duality $\_^\perp$ is transported onto the space $\mathcal{P}\ell_{\mathcal{V}}(X)$ of lenses. However, the resulting notion maps any lens L first to its associated quasi-lens $(\uparrow L, cl(L))$, then swaps the two components, and takes back the union. So duality is just the identity map in this case, blurring the swap between angels and demons:

Corollary 5.24. Let X be a stably compact space. Then $\mathcal{P}\ell_{\mathcal{V}}(X)^{\mathsf{d}} = \mathcal{P}\ell_{\mathcal{V}}(X^{\mathsf{d}})$.

6. The Probabilistic Powerdomain; Mixed Choice I: Games

The notion of *continuous valuation* is a natural alternative to the more well-known notion of measure (Jones, 1990). Instead of defining it directly, we define the more general notion of *game* (Goubault-Larrecq, 2007a). This is modeled after what economists call cooperative games with transferable utility (Gilboa and Schmeidler, 1994), and which take their roots in Gustave Choquet's work on capacities (Choquet, 54). The study of capacities on non-T_2 spaces was initiated by Norberg and Vervaat (Norberg and Vervaat, 1997).

Taking the naming conventions of (Goubault-Larrecq, 2007a), and following (Gilboa and Schmeidler, 1994), a *game* is a strict monotone map ν from $\mathcal{O}(X)$ to $\mathbb{R}^+$; strictness means, as for **A**-valuations, that $\nu(\emptyset) = 0$; monotonicity means that $\nu(U) \leq \nu(V)$ whenever $U \subseteq V$.

Say that the game ν is *modular* (resp., *convex*, resp. *concave*) if and only if $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ (resp. $\geq$, resp. $\leq$) for all opens U, V. The terms supermodular and submodular are sometimes used in lieu of convex, concave. A modular game is called a *valuation*.

A game ν is *totally convex* iff:

$$\nu\left(\bigcup_{i=1}^{n} U_i\right) \geq \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} U_i\right) \quad (1)$$

for every finite family $(U_i)_{i=1}^n$, $n \geq 1$, of opens of X. A *credibility* is a totally convex game. We called credibilities *belief functions* in (Goubault-Larrecq, 2007a), following common usage for credibilities on discrete spaces. The standard name for "totally convex" is "totally monotonic" (Gilboa and Schmeidler, 1994). However, total convexity has a dual that is best named total concavity: ν is *totally concave* iff (1) holds with $\geq$ replaced by $\leq$, and the roles of unions and intersections are swapped. A totally concave game is a *plausibility* (Goubault-Larrecq, 2007a). Note that, if ν is a valuation, then (1) holds with $=$ instead of $\geq$. This equation is the well-known *inclusion-exclusion principle* of probability theory.

For every non-empty subset A of X, the *unanimity game* $\mathfrak{u}_A$ is defined by: $\mathfrak{u}_A(U) = 1$ if $A \subseteq U$, 0 otherwise. We have already introduced this notation in Section 5. Every unanimity game is a credibility (Gilboa and Schmeidler, 1994; Goubault-Larrecq, 2007a). We don't wish to disrupt the flow of exposition, and will prove this later (Lemma 6.15).

A special case of a unanimity game $\mathfrak{u}_A$ is when A is a one-element set $\{x\}$. Then $\mathfrak{u}_{\{x\}}$ is the *Dirac valuation* δ_x at x. Dually, we let the *example game* $\mathfrak{e}_A$ be defined by: $\mathfrak{e}_A(U) = 1$ if $A \cap U \neq \emptyset$, 0 otherwise. Again, we have already introduced this notation in Section 5. Every example game is a plausibility, as we shall see in Lemma 6.22.

Every totally convex game is convex, but the converse fails. E.g., take $X = \{1, 2, 3\}$ with the discrete topology, $\mathfrak{u}_{\{1,2\}}$ is a credibility but not a valuation, and $\frac{1}{2}(\mathfrak{u}_{\{1,2\}} + \mathfrak{u}_{\{1,3\}} + \mathfrak{u}_{\{2,3\}} - \mathfrak{u}_{\{1,2,3\}})$ is a convex game but not a credibility. The latter indeed takes all sets of cardinality 1 or less to 0, all two-element sets to 1/2, and the whole set to 1, from which convexity follows by case analysis. Total convexity fails: take $U_1 = \{2, 3\}$, $U_2 = \{1, 3\}$, $U_3 = \{1, 2\}$, then the left-hand side of (1) is 1, while the right-hand side is $3 \times 1/2$ (the sum of the measures of U_1, U_2, U_3; the other terms contribute zero). Similarly, every totally concave game is concave, but the converse fails.

A *probability valuation* ν (resp., *subprobability valuation*) is a valuation that is *normalized* (resp., *subnormalized*), i.e., that $\nu(X) = 1$ (≤ 1). We shall say *probability* instead of probability valuation, for short.

A game ν is *continuous* iff $\nu(\bigcup_{i\in I} U_i) = \sup_{i\in I} \nu(U_i)$ for every directed family $(U_i)_{i\in I}$ of opens.

Our main focus will be on normalized games. Subnormalized games are not that different: any subnormalized game ν on X extends uniquely to a normalized game $\nu_\perp$ on $X_\perp$ by $\nu_\perp(U) = \nu(U)$ for all opens U of X, and $\nu_\perp(X_\perp) = 1$. Conversely, any normalized game on $X_\perp$ restricts to a unique subnormalized game on X. Moreover, this isomorphism preserves all the properties among continuity, (total) convexity, (total) concavity. This allows us to concentrate on normalized games; corresponding results on subnormalized games easily follow.

Continuous valuations are used to give meaning to probabilistic choice in programming languages (Jones, 1990). Continuous valuations extend to measures on the Borel σ-algebra of the topology, under mild assumptions (Keimel and Lawson, 2005), showing that the two notions are close. In fact, Theorem 8.3 of op.cit. implies that any continuous valuation ν on a stably compact space X extends to a measure on the Borel σ-algebra, not just of X, but even of X^{patch}. (Furthermore, this measure is regular, and is unique among the regular measures extending ν to the Borel σ-algebra of X^{patch}.) Keimel and Lawson consider extended valuations, i.e., maps defined as our valuations above, but with target space $\overline{\mathbb{R}^+}$ instead of $\mathbb{R}^+$, where $\overline{\mathbb{R}^+}$ is $\mathbb{R}^+$ with an added top element $+\infty$, with its Scott topology. Their theorem 8.3 applies to the more general class of locally finite extended valuations. Our valuations ν are always bounded (by $\nu(X)$). Since our interest lies in normalized games, and in fact our duality $\_^\perp$ to come is only defined on normalized games, we won't consider extended valuations.

There is again a topological way and a domain-theoretic way of defining the various spaces of games we shall be interested in. In the domain-theoretic case, we order games by the pointwise ordering: $\nu \leq \nu'$ iff $\nu(U) \leq \nu'(U)$ for every open subset U of X. Let $\mathbf{J}(X)$ be the space of all continuous games on X, $\mathbf{V}(X)$ the space of continuous valuations on X, $\mathbf{Cd}(X)$ that of continuous credibilities on X, $\mathbf{Pb}(X)$ that of continuous plausibilities on X, $\bigtriangledown\mathbf{J}(X)$ the space of continuous convex games on X, and $\bigtriangleup\mathbf{J}(X)$ the space of continuous concave games on X. These are equipped with their Scott topologies. We also add subscripts 1, resp. ≤ 1, to denote their subsets of normalized, resp. subnormalized games, so that, for example, $\mathbf{V}_1(X)$ is the space of continuous probabilities on X.

The topological way is to equip each of these spaces Y with their weak topologies. The *weak topology* on Y is by definition the induced topology from the inclusion of Y in the product $\mathbb{R}^{+\mathcal{O}(X)}_\sigma$, and is generated by the subbasic open sets $[U > r]_Y = \{\nu \in Y \mid \nu(U) > r\}$, U open in X, $r \in \mathbb{R}$. When Y is clear from the context, we shall just write $[U > r]$ for $[U > r]_Y$. Jung calls the weak topology, defined this way, the product topology on $\mathbf{V}_1(X)$ and $\mathbf{V}_{\leq 1}(X)$ (Jung, 2004; Alvarez-Manilla et al., 2004), reserving the name "weak topology" for another definition (see also Definition 7.1). However, the two topologies always coincide.

We append the two letters "wk" in subscript to the various spaces of games above to indicate that we consider them with their weak topology. E.g., $\mathbf{J}_{1\ wk}(X)$ is the space of all continuous normalized games on X with its weak topology, and $\mathbf{V}_{1\ wk}(X)$ and $\mathbf{V}_{\leq 1\ wk}(X)$ are the spaces of continuous probabilities, resp. of continuous subprobabilities, with their weak topologies.

The attentive reader will have noticed that the weak topologies are nothing else than topologies of pointwise convergence, again.

It is well-known that, if X is a continuous dcpo, then so is $\mathbf{V}_{\leq 1}(X)$ (Jones, 1990). A basis is given by the *simple* (subnormalized) valuations $\sum_{i=1}^n a_i \delta_{x_i}$, $a_1, \ldots, a_n \in \mathbb{R}^+$ (and $\sum_{i=1}^n a_i \leq 1$), $x_1, \ldots, x_n \in X$. If X is a continuous *pointed* dcpo, then so is $\mathbf{V}_1(X)$ (Edalat, 1995, Section 3), with a basis of simple normalized valuations. This is an easy consequence of Jones' result, by *Edalat's trick*: observe that $X = Y_\perp$, where Y is the dcpo obtained by removing the least element $\perp$ from X, and Y is again a continuous dcpo with way-below relation obtained by restriction from that of X, so that Jones' result applies to $\mathbf{V}_{\leq 1}(Y) \cong \mathbf{V}_1(X)$.

The Scott topology is in general finer than the weak topology on any space of games. However, the two topologies agree on $\mathbf{V}_{\leq 1}(X)$, i.e., $\mathbf{V}_{\leq 1}(X) = \mathbf{V}_{\leq 1\ wk}(X)$, when X is a continuous dcpo (Tix, 1995, Satz 4.10). By Edalat's trick, they also agree on $\mathbf{V}_1(X)$, i.e., $\mathbf{V}_1(X) = \mathbf{V}_{1\ wk}(X)$, when X is a continuous pointed dcpo.

Jung has shown (Jung, 2004, Theorem 3.2) that whenever X is stably compact, then so are $\mathbf{V}_{\leq 1\ wk}(X)$ and $\mathbf{V}_{1\ wk}(X)$ in their weak topologies. The technique of patch-continuous inequalities of Section 5 was obtained by taking a slightly more abstract view of Jung's technique. It is only right that it applies to sundry spaces of games.

Proposition 6.1. Let X be a topological space, and P be any conjunction of properties of games among "convex", "concave", "totally convex", "totally concave", "modular", "normalized".

The space $J^P(X)$ of all (not necessarily continuous) subnormalized games on X satisfying property P, with the induced topology from the product topology on $[0,1]_\sigma^{\mathcal{O}(X)}$, is stably compact.

Proof. This is as in Proposition 5.6, with $T = \mathcal{O}(X)$, $A = \overline{\mathbb{R}^+}$. Note that $\overline{\mathbb{R}^+}$ is stably compact with its Scott topology. In fact, $\overline{\mathbb{R}^+}$ is a continuous complete lattice, its Scott-open subsets are intervals of the form $(r, +\infty]$, $r \in \mathbb{R}$ (when $r < 0$, we take $(r, +\infty]$ to denote the whole space), its cocompact subsets (which coincide with the opens of the upper topology) are $\overline{\mathbb{R}^+}$ itself, plus all intervals of the form $[0, r)$ where $r \in \overline{\mathbb{R}^+}$.

Note also that $+$ is patch-continuous from $\overline{\mathbb{R}^+} \times \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$, where we define $(+\infty)+y = x + (+\infty) = +\infty$. In fact, $+$ is continuous for both the Scott and the upper topologies. For every $r \in \mathbb{R}$, $(+)^{-1}(r, +\infty] = \bigcup_{\substack{s,t\in\mathbb{R}\\ s+t>r}} (s, +\infty] \times (t, +\infty]$, and for every $r \in \overline{\mathbb{R}^+}$, $(+)^{-1}[0, r) = \bigcup_{\substack{s,t\in\overline{\mathbb{R}^+}\\ s+t<r}} [0, s) \times [0, t)$.

Then $J^P(X)$ is definable as some $[\Sigma^P]$ in each case. Σ^P contains the (in)equations $\_(\emptyset) \doteq 0$ (strictness), $\_(U) \mathrel{\dot{\leq}} \_(V)$ for all opens U and V with $U \subseteq V$ (monotonicity), and $\_(X) \mathrel{\dot{\leq}} 1$ (subnormalized). If convexity is required in P, then we add the inequalities $\_(U) + \_(V) \mathrel{\dot{\leq}} \_(U \cap V) + \_(U \cup V)$ to Σ^P for all opens U, V. The latter in particular is the reason why we did not choose $A = [0,1]_\sigma$, since addition can lead out of $[0,1]$.

The various minus signs in the definition of total convexity seem to pose a difficulty. However, by moving negated terms to the other side of the inequality, we obtain an equivalent formulation using addition only. Namely, if total convexity is required in P, then we add the inequalities $\sum_{\substack{I\subseteq\{1,\ldots,n\}, I\neq\emptyset,\\ |I| \text{ odd}}} \_\left(\bigcap_{i\in I} U_i\right) \mathrel{\dot{\leq}} \_\left(\bigcup_{i=1}^n U_i\right) + \sum_{\substack{I\subseteq\{1,\ldots,n\}, I\neq\emptyset,\\ |I| \text{ even}}} \_\left(\bigcap_{i\in I} U_i\right)$ to Σ^P, for all $n \geq 1$ and all opens $U_1, \ldots, U_n$. We leave the other properties as an exercise. □

The previous proof in fact also shows that the corresponding spaces of *extended* games, i.e., those that may take the value $+\infty$, is also stably compact.

Recall Scott's formula from Lemma 5.7: on games, $\mathfrak{r}(\nu)(U) = \sup_{V \Subset U} \nu(V)$.

Lemma 6.2. Under the assumptions of Proposition 6.1, and provided X is stably compact, for any subnormalized game ν on X satisfying property P, $\mathfrak{r}(\nu)$ is a continuous subnormalized game on X satisfying property P.

Proof. First, $\mathfrak{r}(\nu)$ is clearly a continuous game, and is subnormalized: $\mathfrak{r}(\nu)(X) =$

$\sup_{V \Subset X} \nu(V) \leq 1$. If ν is convex, then:

$$\begin{aligned}
\mathfrak{r}(\nu)(U_1 \cup U_2) + \mathfrak{r}(\nu)(U_1 \cap U_2) &= \sup_{V \Subset U_1 \cup U_2} \nu(V) + \sup_{V \Subset U_1 \cap U_2} \nu(V) \\
&= \sup_{\substack{V_1 \Subset U_1 \\ V_2 \Subset U_2}} \nu(V_1 \cup V_2) + \sup_{\substack{V_1 \Subset U_1 \\ V_2 \Subset U_2}} \nu(V_1 \cap V_2) \\
&\qquad \text{since } \cup \text{ (core-compactness) and } \cap \text{ (core-coherence)} \\
&\qquad \text{preserve and reflect } \Subset \\
&= \sup_{\substack{V_1 \Subset U_1 \\ V_2 \Subset U_2}} (\nu(V_1 \cup V_2) + \nu(V_1 \cap V_2))
\end{aligned}$$

since $+$ is Scott-continuous on $\mathbb{R}^+_\sigma$. Note that the sups are directed, since $\mathcal{O}(X)$, hence $\mathcal{O}(X) \times \mathcal{O}(X)$, is a continuous dcpo. Since ν is convex, this is greater than or equal to:

$$\begin{aligned}
\sup_{\substack{V_1 \Subset U_1 \\ V_2 \Subset U_2}} (\nu(V_1) + \nu(V_2)) &= \sup_{\substack{V_1 \Subset U_1 \\ V_2 \Subset U_2}} \nu(V_1) + \sup_{\substack{V_1 \Subset U_1 \\ V_2 \Subset U_2}} \nu(V_2) \\
&= \sup_{V_1 \Subset U_1} \nu(V_1) + \sup_{V_2 \Subset U_2} \nu(V_2) = \mathfrak{r}(\nu)(U_1) + \mathfrak{r}(\nu)(U_2)
\end{aligned}$$

So $\mathfrak{r}(\nu)$ is convex, too. The proof that $\mathfrak{r}(\nu)$ is concave, resp. modular, whenever ν is, is similar.

If ν is totally convex, we show that $\mathfrak{r}(\nu)$ is totally convex as above. Remember that total convexity (for $\mathfrak{r}(\nu)$) has the following equivalent formulation with no minus sign, a fact that we have already used in Proposition 6.1:

$$\mathfrak{r}(\nu)\left(\bigcup_{i=1}^n U_i\right) + \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset, \\ |I| \text{ even}}} \mathfrak{r}(\nu)\left(\bigcap_{i \in I} U_i\right) \quad \geq \quad \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset, \\ |I| \text{ odd}}} \mathfrak{r}(\nu)\left(\bigcap_{i \in I} U_i\right)$$

This is proved as convexity above, and is left as an exercise. Similarly, if ν is totally concave, then so is $\mathfrak{r}(\nu)$.

Finally, if ν is normalized, then note that X, as an open set, is such that $X \Subset X$, as X is compact. So $\mathfrak{r}(\nu)(X) = \sup_{V \Subset X} \nu(V) \geq \nu(X) = 1$, hence $\mathfrak{r}(\nu)$ is normalized. □

Lemma 6.3. Under the assumptions of Proposition 6.1, and provided X is stably compact, $\mathfrak{r}$ is continuous from $J^P(X)$ to the space $\mathbf{J}^P_{wk}(X)$ of continuous subnormalized games on X satisfying P, with the weak topology, and forms a retraction with the canonical inclusion $\mathfrak{s}$ as associated section.

Proof. Use Lemma 5.10, with $Z = J^P(X)$, $Y = \mathbb{R}^+$. One has $\mathfrak{r}(Z) \subseteq Z$ by Lemma 6.2, and $\mathfrak{r}(Z)$, as a set, coincides with $\mathbf{J}^P(X)$. □

Using Lawson's Lemma 5.12, it follows:

Proposition 6.4. If X is stably compact, then so are the spaces $\mathbf{J}_{\leq 1\ wk}(X)$, $\bigtriangledown \mathbf{J}_{\leq 1\ wk}(X)$, $\bigtriangleup \mathbf{J}_{\leq 1\ wk}(X)$, $\mathbf{Cd}_{\leq 1\ wk}(X)$, $\mathbf{Pb}_{\leq 1\ wk}(X)$, $\mathbf{V}_{\leq 1\ wk}(X)$, as well as $\mathbf{J}_{1\ wk}(X)$, $\bigtriangledown \mathbf{J}_{1\ wk}(X)$, $\bigtriangleup \mathbf{J}_{1\ wk}(X)$, $\mathbf{Cd}_{1\ wk}(X)$, $\mathbf{Pb}_{1\ wk}(X)$, $\mathbf{V}_{1\ wk}(X)$.

Jung's results, cited above, were that $\mathbf{V}_{\leq 1\ wk}(X)$ and $\mathbf{V}_{1\ wk}(X)$ are stably compact for every stably compact space X. As we have said earlier, for spaces of continuous valuations, the above *is* Jung's proof (Jung, 2004; Alvarez-Manilla et al., 2004). The technique can be considered to have its root in the proof of the Banach-Alaoglu Theorem. See also Plotkin's proof of a domain-theoretic variant of the latter (Plotkin, 2006), also inspired by Jung.

We now characterize the de Groot dual of these spaces. As for **A**-valuations, this goes through the definition of a dagger operation $\_^\dagger$. This was defined for continuous valuations in (Tix, 1995).

Definition 6.5. Let X be a topological space. Given any continuous game ν, let $\nu^\dagger(Q)$ be defined, for every compact saturated subset Q of X, by:

$$\nu^\dagger(Q) = \inf_{\substack{U \in \mathcal{O}(X)\\ Q \subseteq U}} \nu(U).$$

Lemma 6.6. Let X be stably compact, and Y be any of the spaces of continuous games of Proposition 6.4.

For every compact saturated subset Q of X, for every $r \in \mathbb{R}$, define $\langle Q \geq r\rangle = \{\nu \in Y \mid \nu^\dagger(Q) \geq r\}$; then $\langle Q \geq r\rangle$ is compact saturated in Y.

Proof. This is as for Lemma 5.15. In each case, Y is the space $\mathbf{J}^P_{wk}(X)$ for some conjunction of properties P as in Proposition 6.1, and is definable as $[\Sigma^P]$ for some system of patch-continuous inequalities Σ^P. Consider $[\Sigma^P_{Q\geq r}]$, where $\Sigma^P_{Q\geq r}$ is obtained from Σ^P by adding all patch-continuous inequalities $r \stackrel{\cdot}{\leq} \_(U)$, for all opens U of X containing Q. By Proposition 5.5, $[\Sigma^P_{Q\geq r}]$ is a stably compact space with the topology induced by the product topology on $[0,1]_\sigma^{\mathcal{O}(X)}$.

Now $\mathfrak{r}$ is not only continuous from $J^P(X) = [\Sigma^P]$ to $\mathbf{J}^P_{wk}(X)$, but also from $[\Sigma^P_{Q\geq r}]$ to $\langle Q \geq r\rangle$, seen as a subspace of $\mathbf{J}^P_{wk}(X)$ with the induced topology (i.e., with the weak topology). This is again by Lemma 5.10. We must check that for every $\nu \in [\Sigma^P_{Q\geq r}]$, for every open U of X containing Q, $\mathfrak{r}(\nu)(U) \geq r$: note that U is the directed union of all opens $V \Subset U$, and since Q is compact and contained in U, there must be an open V such that $Q \subseteq V \Subset U$. By assumption $\nu(V) \geq r$, so $\mathfrak{r}(\nu)(U) = \sup_{V \Subset U} \nu(V) \geq r$.

Clearly, $\mathfrak{r}$ is a retraction, with canonical inclusion as section, so $\langle Q \geq r\rangle$ is a retract of $[\Sigma^P_{Q\geq r}]$. By Lawson's Lemma 5.12, it is a stably compact space. As a subset of $\mathbf{J}^P_{wk}(X)$, it is then compact. That it is saturated is clear. ☐

Lemma 6.7. Let X be a topological space, and Y be any space of maps from $\mathcal{O}(X)$ to $[0,1]_\sigma$. Equip Y with its weak topology. Then the specialization ordering of Y is the pointwise ordering: $\nu \leq \nu'$ iff $\nu(U) \leq \nu'(U)$ for all opens U of X.

Proof. Write temporarily $\preceq$ for the specialization ordering, $\leq$ for the pointwise ordering. Recall that $\nu \preceq \nu'$ iff ν' belongs to any weak open that contains ν. Equivalently, iff whenever $\nu \in [U > r]$, then $\nu' \in [U > r]$, U open in X, $r \in \mathbb{R}$. So, if $\nu \leq \nu'$, then $\nu \preceq \nu'$. Conversely, if $\nu \preceq \nu'$, for every open U and every real $r < \nu(U)$, we clearly have $\nu \in [U > r]$, so $\nu' \in [U > r]$, i.e., $\nu'(U) > r$. Taking the sup over all values of r, $\nu(U) \leq \nu'(U)$. ☐

Proposition 6.8. Let X be stably compact, and Y be any of the spaces of continuous games of Proposition 6.4. The compact saturated subsets of Y are exactly the intersections of finite unions of sets of the form $\langle Q \geq r \rangle$, Q compact saturated in X, $r \in \mathbb{R}$. In other words, the topology of Y^{d} is generated by complements of sets of the form $\langle Q \geq r \rangle$, Q compact saturated in X, $r \in \mathbb{R}$.

Proof. This is similar to Proposition 5.18. One direction is Lemma 6.6. Conversely, let $\mathcal{Q}$ be a compact saturated subset of Y. As a subset of $[\mathcal{O}(X) \to [0,1]_\sigma]_{\mathsf{p}}$, $\mathcal{Q}$ is again compact, since the weak topology on Y is induced from the product topology. Write $\uparrow \mathcal{Q}$ the upward-closure of $\mathcal{Q}$ in $[\mathcal{O}(X) \to [0,1]_\sigma]_{\mathsf{p}}$. This is compact saturated in $[\mathcal{O}(X) \to [0,1]_\sigma]_{\mathsf{p}}$. Now $[0,1]_\sigma$ is a bc-domain with least element, and X is locally compact, so $[\mathcal{O}(X) \to [0,1]_\sigma]_{\mathsf{p}} = [\mathcal{O}(X) \to [0,1]_\sigma]$ by Lemma 5.16. So $\uparrow \mathcal{Q}$ is compact saturated in $[\mathcal{O}(X) \to [0,1]_\sigma]$, and Lemma 5.17 implies that $\uparrow \mathcal{Q}$ is an intersection of finite unions of sets of the form $\langle Q \geq r \rangle^*$, Q compact saturated subset of X, $r \in [0,1]$. Since $\mathcal{Q}$ is saturated in Y, and since the specialization orderings of Y and of $[\mathcal{O}(X) \to [0,1]_\sigma]_{\mathsf{p}}$ are the same (Lemma 6.7), $\mathcal{Q} = \uparrow \mathcal{Q} \cap Y$. So $\mathcal{Q}$ is the intersection of finite unions of sets of the form $\langle Q \geq r \rangle^* \cap Y = \langle Q \geq r \rangle$. □

The above result was claimed in (Jung, 2004, last lines) for spaces of continuous (sub)normalized valuations.

Definition 6.9 (Dual). Let X be a stably compact space. For every normalized game ν on X, let the *dual* $\nu^\perp$ of ν be the map from $\mathcal{O}(X^{\mathsf{d}})$ to $\mathbb{R}^+_\sigma$ defined by:

$$\nu^\perp(X \setminus Q) = 1 - \nu^\dagger(Q).$$

This is of course very similar to Definition 5.19.

The definition may seem overly restrictive: we require ν to be normalized. One might think of generalizing this to any game by letting $\nu^\perp(X \setminus Q) = \nu(X) - \nu^\dagger(Q)$. However, if duality is to work on games, then in particular $\nu \leq \nu'$ should imply $\nu'^\perp \leq \nu^\perp$—de Groot duality reverses order—and this fails with the relaxed definition. Indeed, consider $X = \{1, 2, \perp\}$, where $\perp$ is least, and 1 and 2 are incomparable. Let $\nu = \frac{1}{2}\mathfrak{u}_X$, $\nu' = \frac{1}{4}\delta_1 + \frac{3}{4}\delta_2$. One checks that $\nu \leq \nu'$ on the four non-empty opens of X: $\nu\{1\} = \nu\{2\} = \nu\{1,2\} = 0$, while $\nu(X) = \frac{1}{2}$, $\nu'(X) = 1$. But $\nu^\perp$ and $\nu'^\perp$ are incomparable: e.g., $\nu^\perp$ maps the four non-empty cocompacts $\{\perp\}$, $\{\perp, 1\}$, $\{\perp, 2\}$ and X to $\frac{1}{2}$, while $\nu'^\perp$ maps them to 0, $\frac{1}{4}$, $\frac{3}{4}$, and 1 respectively. So $\_^\perp$, with the relaxed definition would not be monotonic from $\mathbf{J}_{\leq 1\ wk}(X)$ to $\mathbf{J}_{\leq 1\ wk}(X^{\mathsf{d}})^{\mathsf{d}}$. One may object to this example, and prefer one with no unanimity game and only continuous valuations. Using the one-to-one correspondence between credibilities on X and continuous valuations on $\mathcal{Q}(X)$ (Theorem 6.18 below) suggests considering $\mathcal{Q}(X)$, which is a four-element domain, and one would take $\nu^* = \frac{1}{2}\delta_X$, $\nu'^* = \frac{1}{4}\delta_{\{1\}} + \frac{3}{4}\delta_{\{2\}}$. We let the reader check that $\nu^* \leq \nu'^*$, but $\nu^{*\perp}$ and $\nu'^{*\perp}$ are incomparable.

Up to the use of $\_^\perp$ instead of $\_^\dagger$, and up to the fact that we are considering games and not just valuations, the following is (Tix, 1995, Satz 3.4).

Lemma 6.10 (Convex-concave Duality). Let X be stably compact. For every continuous normalized game ν on X, $\nu^\perp$ is a continuous normalized game on X^{d}. If ν is (totally) convex, then $\nu^\perp$ is (totally) concave. If ν if (totally) concave, then $\nu^\perp$ is (totally) convex. If ν is a valuation, then so is $\nu^\perp$. Finally, $\nu^{\perp\perp} = \nu$.

Proof. Strictness, normalization, monotonicity, continuity, and the fact that $\nu^{\perp\perp} = \nu$ are as in Lemma 5.20, modulo change of notation. For every finite collection $Q_1, Q_2, \ldots, Q_n$ of compact saturated subsets of X ($n \geq 1$),

$$\nu^\dagger(\bigcup_{i=1}^n Q_i) \quad = \quad \inf_{U \supseteq \bigcup_{i=1}^n Q_i} \nu(U) = \inf_{U_i \supseteq Q_i \text{ for all } i, 1 \leq i \leq n} \nu(\bigcup_{i=1}^n U_i)$$

Indeed, the $\geq$ direction is by choosing $U_i = U$ for all i, and the $\leq$ direction is by monotonicity. We also have:

$$\nu^\dagger(\bigcap_{i=1}^n Q_i) \quad = \quad \inf_{U \supseteq \bigcap_{i=1}^n Q_i} \nu(U) = \inf_{U_i \supseteq Q_i \text{ for all } i, 1 \leq i \leq n} \nu(\bigcap_{i=1}^n U_i)$$

The $\leq$ direction is again by monotonicity, while the $\geq$ direction is because X is weakly Hausdorff. It follows that, if ν is convex, resp. totally convex, resp. concave, resp. totally concave, then so is $\nu^\dagger$. We deal with total convexity to demonstrate how this works. First, note that the families of opens $(U_i)_{i=1}^n$ such that $U_i \supseteq Q_i$ for all i, $1 \leq i \leq n$, are filtered in the n-fold product of the inclusion ordering. Then, addition is *co-continuous* on $\mathbb{R}^+$, i.e., commutes with infs of filtered families. So:

$$\begin{aligned}
& \nu^\dagger(\bigcup_{i=1}^n Q_i) + \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ even}}} \nu^\dagger(\bigcap_{i \in I} Q_i) \\
= \quad & \inf_{U_i \supseteq Q_i \text{ for all } i, 1 \leq i \leq n} \nu(\bigcup_{i=1}^n U_i) + \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ even}}} \inf_{U_i \supseteq Q_i \text{ for all } i, 1 \leq i \leq n} \nu(\bigcap_{i \in I} U_i) \\
= \quad & \inf_{U_i \supseteq Q_i \text{ for all } i, 1 \leq i \leq n} \left(\nu(\bigcup_{i=1}^n U_i) + \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ even}}} \nu(\bigcap_{i \in I} U_i) \right) \\
\geq \quad & \inf_{U_i \supseteq Q_i \text{ for all } i, 1 \leq i \leq n} \left(\sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ odd}}} \nu(\bigcap_{i \in I} U_i) \right) \\
= \quad & \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ odd}}} \inf_{U_i \supseteq Q_i \text{ for all } i, 1 \leq i \leq n} \nu(\bigcap_{i \in I} U_i) = \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ odd}}} \nu^\dagger \left(\bigcap_{i \in I} Q_i \right)
\end{aligned}$$

It follows that $\_^\perp$ exchanges (total) convexity with (total) concavity. Since valuations are just games that are both convex and concave, $\_^\perp$ maps valuations to valuations. □

Theorem 6.11 (Duality, Games, Topological Version). Let X be a stably compact space. Then $\_^\perp$ defines an involutive homeomorphism, hence also an order-isomorphism:

— from $\mathbf{J}_{1\ wk}(X)^{\mathsf{d}}$ to $\mathbf{J}_{1\ wk}(X^{\mathsf{d}})$;
— from $\bigtriangledown\mathbf{J}_{1\ wk}(X)^{\mathsf{d}}$ to $\bigtriangleup\mathbf{J}_{1\ wk}(X^{\mathsf{d}})$;
— from $\bigtriangleup\mathbf{J}_{1\ wk}(X)^{\mathsf{d}}$ to $\bigtriangledown\mathbf{J}_{1\ wk}(X^{\mathsf{d}})$;
— from $\mathbf{Cd}_{1\ wk}(X)^{\mathsf{d}}$ to $\mathbf{Pb}_{1\ wk}(X^{\mathsf{d}})$;
— from $\mathbf{Pb}_{1\ wk}(X)^{\mathsf{d}}$ to $\mathbf{Cd}_{1\ wk}(X^{\mathsf{d}})$;
— from $\mathbf{V}_{1\ wk}(X)^{\mathsf{d}}$ to $\mathbf{V}_{1\ wk}(X^{\mathsf{d}})$.

Proof. Let P be any property among "convex", "concave", "totally convex", "totally concave", "modular", and true. Let $\overline{P}$ be the property obtained from P by exchanging (totally) "convex" with (totally) "concave". Then $\_^{\perp}$ maps $\mathbf{J}^{\overline{P}}_{wk}(X)^{\mathsf{d}}$ to $\mathbf{J}^{P}_{wk}(X^{\mathsf{d}})$ by Lemma 6.10. The inverse image by $\_^{\perp}$ of the subbasic open $[X \setminus Q > r]$ of $\mathbf{J}^{P}_{wk}(X^{\mathsf{d}})$ (Q compact saturated in X, $r \in \mathbb{R}$) is the complement of $\langle Q \geq 1 - r\rangle$, and the direct image of the complement of $\langle Q \geq r\rangle$ is $[X \setminus Q > 1 - r]$. By Proposition 6.8, $\_^{\perp}$ is continuous and open from $\mathbf{J}^{\overline{P}}_{wk}(X)^{\mathsf{d}}$ to $\mathbf{J}^{P}_{wk}(X^{\mathsf{d}})$. By the last claim of Lemma 6.10, $\_^{\perp}$ is involutive, hence a homeomorphism. □

In the special case of continuous valuations, $\mathbf{V}_1(X)^{\mathsf{d}}$ and $\mathbf{V}_1(X^{\mathsf{d}})$ are homeomorphic, justifying our claim that nature is invariant under duality. This much can be proved using an alternative, partly measure-theoretic argument: continuous valuations ν on stably compact spaces X have unique extensions to regular Borel measures on X^{patch} (Keimel and Lawson, 2005, Theorem 8.3). Call μ this extension. The construction in op. cit. shows that $\mu(Q) = \nu^{\dagger}(Q)$ for any compact saturated subset Q of X, whence $\mu(X \setminus Q)$ coincides with our $\nu^{\perp}(X \setminus Q)$. Since μ is regular, $\nu^{\perp}$ is continuous. Moreover, one can show that the topologies are the right ones in each case, using the fact that they coincide with the topology induced by the weak topology on Borel measures on X^{patch} induced by the functionals $\mu \mapsto \int_{x \in X^{\mathsf{patch}}} f(x) d\mu$, where f ranges over the perfect maps from X to $[0,1]_{\sigma}$—i.e., the patch-continuous, order-preserving maps—see (Alvarez-Manilla et al., 2004, Theorem 36).

Corollary 6.12 (Duality, Games, Domain-Theoretic). Let X be a stably compact space such that both X and X^{d} are dcpos—e.g., a stably bicontinuous bicpo. Then $\_^{\perp}$ defines an involutive order-isomorphism:

— from $\mathbf{J}_1(X)^{\mathsf{op}}$ to $\mathbf{J}_1(X^{\mathsf{op}})$;
— from $\bigtriangledown\mathbf{J}_1(X)^{\mathsf{op}}$ to $\bigtriangleup\mathbf{J}_1(X^{\mathsf{op}})$;
— from $\bigtriangleup\mathbf{J}_1(X)^{\mathsf{op}}$ to $\bigtriangledown\mathbf{J}_1(X^{\mathsf{op}})$;
— from $\mathbf{Cd}_1(X)^{\mathsf{op}}$ to $\mathbf{Pb}_1(X^{\mathsf{op}})$;
— from $\mathbf{Pb}_1(X)^{\mathsf{op}}$ to $\mathbf{Cd}_1(X^{\mathsf{op}})$;
— from $\mathbf{V}_1(X)^{\mathsf{op}}$ to $\mathbf{V}_1(X^{\mathsf{op}})$.

Proof. We deal with the first claim, the others are similar. Since $\_^{\perp}$ is a homeomorphism from $\mathbf{J}_{1\ wk}(X)^{\mathsf{d}}$ to $\mathbf{J}_{1\ wk}(X^{\mathsf{d}})$, it is an order-isomorphism between the underlying posets, i.e., from $\mathbf{J}_1(X)^{\mathsf{op}}$ to $\mathbf{J}_1(X^{\mathsf{d}})$. Then, since X^{d} is a dcpo, it has the Scott topology of X^{op}, i.e., $X^{\mathsf{d}} = X^{\mathsf{op}}$. □

One may also observe that $\_^{\perp}$ reverses order directly: if $\nu \leq \nu'$, then $\nu^{\dagger} \leq {\nu'}^{\dagger}$, by the definition of $\dagger$, hence ${\nu'}^{\perp} \leq \nu^{\perp}$.

The next corollary requires the following auxiliary lemma first.

Lemma 6.13. Let X be a stably compact space. For any continuous games ν, ν' on X, for any $r \in [0,1]$, $(r\nu + (1-r)\nu')^\perp = r\nu^\perp + (1-r)\nu'^\perp$.

The dual of any simple probability valuation $\sum_{i=1}^n a_i \delta_{x_i}$ on X is the simple probability valuation $\sum_{i=1}^n a_i \delta_{x_i}$ on X^{d}.

Proof. The first claim follows easily from the fact that $(r\nu)^\dagger = r\nu^\dagger$, $(\nu + \nu')^\dagger = \nu^\dagger + \nu'^\dagger$, which follow from the co-continuity of multiplication by a non-negative real and of addition respectively. (A map is *co-continuous* iff it commutes with filtered infs; we have already used the notion in Lemma 6.10.) The second claim follows from the first and the fact that $\delta_x^\perp = \delta_x$. Indeed, for any open $X \setminus Q$ of X^{d}, $\delta_x^\perp(X \setminus Q) = 1$ iff there is an open U containing Q such that $\delta_x(U) = 0$, iff there is an open U containing Q that does not contain x. Since Q is saturated, this is equivalent to $x \notin Q$, i.e., $\delta_x(X \setminus Q) = 1$. (The fact that $\delta_x^\perp = \delta_x$ will be generalized in Lemma 6.21.) ☐

Corollary 6.14. For any stably bicontinuous bicpo X with a bottom $(\bot)$ and a top $(\top)$ element, $\mathbf{V}_1(X)$ is a stably bicontinuous bicpo with bottom element $\delta_\bot$ and top element $\delta_\top$, and $\_^\perp$ is an involutive homeomorphism from $\mathbf{V}_1(X)^{\mathsf{op}}$ to $\mathbf{V}_1(X^{\mathsf{op}})$.

Every continuous probability ν on X is both the least upper bound of a directed family of simple probabilities way-below ν, and the inf of a filtered family of simple probabilities way-above ν.

Proof. The first part follows from the fact that $\mathbf{V}_1(Z)$ is a continuous dcpo with bottom $\delta_\bot$ as soon as Z is a continuous dcpo with bottom $\bot$ (Edalat, 1995, Section 3), that the Scott topology agrees with the weak topology, and from Corollary 6.12. The second part follows from Lemma 6.13, second part. ☐

Again, the way-below and the converse of the way-above relations do not coincide in general. Take $X = \{\bot, \top\}$ with $\bot < \top$, for example, then $\mathbf{V}_1(X)$ is isomorphic to $[0,1]_\sigma$.

Continuous probability valuations clearly encode probabilistic choice, with no non-determinism. We now demonstrate that credibilities encode certain mixes of demonic non-determinism with probabilistic choice. This will have to be done by hand. However, convex-concave duality will allow us to conclude immediately that plausibilities encode exactly the corresponding mix of angelic non-determinism with probabilistic choice.

It is time we proved that every unanimity game was indeed a credibility.

Lemma 6.15. Every unanimity game $\mathfrak{u}_A$ $(A \neq \emptyset)$ is a normalized credibility. It is continuous iff A is compact, and $\mathfrak{u}_A = \mathfrak{u}_{\uparrow A}$.

Proof. The argument rests on the well-known fact that if $n \geq 1$, for any subset $I_0 \subseteq \{1, \ldots, n\}$, the sum $\sum_{\substack{I \subseteq I_0 \\ I \neq \emptyset}} (-1)^{|I|+1}$ is 0 if I_0 is empty, and 1 otherwise, which we shall call the *Moebius identity*.

Let I_0 be set of all indices i, $1 \leq i \leq n$, such that $A \subseteq U_i$. Then $\mathfrak{u}_A(\bigcap_{i \in I} U_i)$ is 1 if $I \subseteq I_0$, 0 otherwise. To prove (1), it is therefore enough to show $\mathfrak{u}_A\left(\bigcup_{i=1}^n U_i\right) \geq$

$\sum_{\substack{I \subseteq I_0 \\ I \neq \emptyset}} (-1)^{|I|+1}$. If $I_0 = \emptyset$, then the right-hand side is 0. Otherwise, both sides of the inequality equal 1.

If A is compact, then for any directed family $(U_i)_{i \in I}$ of opens, $\mathfrak{u}_A(\bigcup_{i \in I} U_i)$ equals 1 iff $A \subseteq \bigcup_{i \in I} U_i$, iff $A \subseteq U_i$ for some $i \in I$, iff $\sup_{i \in I} \mathfrak{u}_A(U_i) = 1$. Conversely, if $\mathfrak{u}_A$ is continuous, any cover of A by a directed family $(U_i)_{i \in I}$ of opens is such that $1 = \mathfrak{u}_A(\bigcup_{i \in I} U_i) = \sup_{i \in I} \mathfrak{u}_A(U_i)$, so $A \subseteq U_i$ for some $i \in I$, whence A is compact.

The fact that $\mathfrak{u}_A = \mathfrak{u}_{\uparrow A}$ is clear. $\square$

So the only continuous unanimity games are of the form $\mathfrak{u}_Q$ where Q is in the Smyth powerdomain of X. This goes one step further:

Proposition 6.16. Let X be well-filtered and locally compact. The map $\mathfrak{u} : Q \mapsto \mathfrak{u}_Q$ is a continuous order-embedding of $\mathcal{Q}(X)$ into $\mathbf{Cd}_1(X)$, and a topological embedding of $\mathcal{Q}(X)$ into $\mathbf{Cd}_{1\ wk}(X)$.

Proof. That $\mathfrak{u}$ is monotonic is clear. Continuity follows from the fact that, for every filtered family $(Q_i)_{i \in I}$ of non-empty compact saturated subsets of X, for every open U, $\mathfrak{u}_{\bigcap_{i \in I} Q_i}(U) = 1$ iff $\bigcap_{i \in I} Q_i \subseteq U$, iff $Q_i \subseteq U$ for some $i \in I$ by well-filteredness, iff $\sup_{i \in I} \mathfrak{u}_{Q_i}(U) = 1$. The fact that $\mathfrak{u}$ is an order-embedding means that $Q \supseteq Q'$ iff $\mathfrak{u}_Q \leq \mathfrak{u}_{Q'}$. If $\mathfrak{u}_Q \leq \mathfrak{u}_{Q'}$, then every open U that contains Q is such that $\mathfrak{u}_Q(U) = 1$, hence $\mathfrak{u}_{Q'}(U) = 1$, so U contains Q'. Since Q is saturated, we conclude that it contains Q'.

Now $\mathfrak{u}$ is also continuous from $\mathcal{Q}(X)$ to $\mathbf{Cd}_{1\ wk}(X)$, since $\mathfrak{u}^{-1}[U > r]$ equals $\Box U$ if $0 \leq r < 1$, is empty if $r \geq 1$, and the whole of $\mathcal{Q}(X)$ if $r < 0$. It is an embedding because the image of $\Box U$ is, say, $[U > 1/2] \cap \mathbf{Cd}_1(X)$, and such sets $\Box U$ generate the topology of $\mathcal{Q}(X)$ when X is locally compact. $\square$

Clearly, there is also a continuous order-embedding of $\mathbf{V}_1(X)$ into $\mathbf{Cd}_1(X)$ which is also a topological embedding of $\mathbf{V}_{1\ wk}(X)$ into $\mathbf{Cd}_{1\ wk}(X)$: the canonical inclusion. We take this is as (relatively weak, for now) evidence that credibilities encode both demonic and probabilistic choice.

We have already introduced $\mathfrak{u}_Q$ in Section 5, where these served as a functional description of the Smyth powerdomain. One can make this more precise, and give an explicit characterization of those elements in the image of the embedding $\mathfrak{u}$.

Proposition 6.17. Let X be a sober space. Any continuous normalized credibility ν such that ν only takes values 0 or 1 is of the form $\mathfrak{u}_Q$, $Q \in \mathcal{Q}(X)$. In fact, this already holds of all normalized convex games that take values 0 or 1 only.

Proof. Let ν be a convex game, $\nu(X) = 1$, and ν only takes values 0 or 1. Let $\mathcal{F}$ be the collection of all opens U such that $\nu(U) = 1$. This is a Scott-open filter of opens. Moreover, $\mathcal{F}$ is non-trivial, i.e., not the whole of $\mathcal{O}(X)$, since $\emptyset \notin \mathcal{F}$. The Hofmann-Mislove Theorem implies that the intersection Q of all elements of $\mathcal{F}$ is a non-empty compact saturated subset of X, and that $Q \subseteq U$ iff $U \in \mathcal{F}$, whence $\nu = \mathfrak{u}_Q$. $\square$

The fundamental theorem of credibilities is (Goubault-Larrecq, 2007a, Theorem 1), which states similar evidence with much stronger force: essentially, it states that continuous credibilities are nothing else than specifications of random choices among sets

Q of possible demonic choices, i.e., they are continuous valuations on $\mathcal{Q}(X)$. (This is a refinement of the "completion of a misspecified model" of (Gilboa and Schmeidler, 1994, Section 5).) We restate it below, together with the statement of Theorem 2 (Goubault-Larrecq, 2007a). It turns out that Theorem 3 and Lemma 1 of op. cit. are wrong, see Proposition 6.19 below. The proofs of the following results can be found in the companion paper (Goubault-Larrecq, 2009, Section 3).

Theorem 6.18. Let X be a topological space. For any continuous valuation P on $\mathcal{Q}(X)$, the game $P_\Box$ defined by $P_\Box(U) = P(\Box U)$ is a continuous credibility on X.

The map $P \mapsto P_\Box$ is monotonic, Scott-continuous, and continuous from $\mathbf{V}_{1\ wk}(\mathcal{Q}(X))$ to $\mathbf{Cd}_{1\ wk}(X)$.

Assume now that X is well-filtered and locally compact space. Then, for every continuous credibility ν on X there is a unique continuous valuation ν^* on $\mathcal{Q}(X)$ such that $\nu(U) = \nu^*(\Box U)$ for every open U of X.

The map $\nu \mapsto \nu^*$ is a one-to-one mapping from $\mathbf{Cd}_1(X)$ to $\mathbf{V}_1(\mathcal{Q}(X))$, whose inverse is $P \mapsto P_\Box$.

If additionally X is compact, then $\mathbf{Cd}_1(X)$ is a continuous dcpo with bottom element $\mathfrak{u}_X$, and with basis given by *simple credibilities* $\sum_{i=1}^n a_i \mathfrak{u}_{Q_i}$ ($a_1, \ldots, a_n \in \mathbb{R}^+$, $Q_1, \ldots, Q_n \in \mathcal{Q}(X)$) that are normalized ($\sum_{i=1}^n a_i = 1$).

Moreover, $\left(\sum_{i=1}^n a_i \mathfrak{u}_{Q_i}\right)^* = \sum_{i=1}^n a_i \delta_{Q_i}$.

More precisely, we have the stronger result that any continuous normalized credibility ν on X is the least upper bound of a family of simple normalized credibilities $(\nu_i)_{i \in I}$ way-below ν, such that $(\nu_i^*)_{i \in I}$ is also a directed family of simple probabilities way-below ν^* in $\mathbf{V}_1(\mathcal{Q}(X))$.

Note that $\nu \mapsto \nu^*$ is a one-to-one mapping, but is *not* an isomorphism in general. In fact, it is not continuous, and not even monotonic. This explains the complex form that the last statement of the above Theorem must have.

Proposition 6.19. The mapping $\nu \mapsto \nu^*$ is not monotonic in general from $\mathbf{Cd}_1(X)$ to $\mathbf{V}_1(\mathcal{Q}(X))$, even when X is finite.

Proof. Consider the space $X = \{1, 2, \top\}$, where $1 \leq \top$, $2 \leq \top$, and 1 and 2 are incomparable, in its Scott topology. Let $\nu = \frac{1}{2}\mathfrak{u}_{\{1,\top\}} + \frac{1}{2}\mathfrak{u}_{\{2,\top\}}$, and $\nu' = \frac{1}{2}\mathfrak{u}_{\{\top\}} + \frac{1}{2}\mathfrak{u}_{\{1,2,\top\}}$. Then $\nu \leq \nu'$, as one easily checks. But $\nu^* \not\leq \nu'^*$, since $\nu^*(\{\{1,\top\},\{2,\top\},\{\top\}\}) = 1$, but $\nu'^*(\{\{1,\top\},\{2,\top\},\{\top\}\}) = \frac{1}{2}$. ☐

Another way to put it is as follows: there is *another* topology on $\mathbf{Cd}_1(X)$ that is induced from the weak topology on $\mathbf{V}_1(\mathcal{Q}(X))$ by $\nu \mapsto \nu^*$. This is strictly finer than the weak topology on $\mathbf{Cd}_1(X)$.

Proposition 6.20 (Soft Topology). Let X be stably compact. Let the *soft topology* on $\mathbf{Cd}_1(X)$ be induced from the weak topology on $\mathbf{V}_1(\mathcal{Q}(X))$ by $\nu \mapsto \nu^*$, i.e., the topology whose opens are of the form $\{\nu \in \mathbf{Cd}_1(X) \mid \nu^* \in \mathcal{U}\}$, where $\mathcal{U}$ ranges over the open subsets of $\mathbf{V}_{1\ wk}(\mathcal{Q}(X))$.

The soft topology is finer than the weak topology, in general strictly even on finite spaces. It is generated by the subbasic opens of the form:

$$[\sum^{\pm} U_1, \ldots, U_n > r] \quad = \quad \left\{ \nu \in \mathbf{Cd}_1(X) \mid \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu(\bigcap_{i \in I} U_i) > r \right\}$$

where $n \geq 1$, $r \in \mathbb{R}$, and $U_1, \ldots, U_n$ are open in X. Its specialization ordering $\preceq$ is defined by $\nu \preceq \nu'$ iff, for every $n \geq 1$, for all opens $U_1, \ldots, U_n$,

$$\sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu(\bigcap_{i \in I} U_i) \quad \leq \quad \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu'(\bigcap_{i \in I} U_i)$$

Proof. Call *elementary open* of $\mathcal{Q}(X)$ any finite union of basic open subsets $\Box U$, U open in X. Every open subset of $\mathcal{Q}(X)$ is a directed union of elementary opens.

The soft topology is generated by the inverse images of subbasic opens $[\mathcal{U} > r]$ by the map $\nu \mapsto \nu^*$, where $\mathcal{U}$ ranges over the open subsets of $\mathcal{Q}(X)$. There is no need to consider the case where $\mathcal{U}$ is empty, since $[\emptyset > r]$ is either empty or the whole of $\mathbf{V}_1(\mathcal{Q}(X))$. Since all games considered here are continuous, we do not change the topology by requiring $\mathcal{U}$ to be elementary. Let therefore write $\mathcal{U}$ as $\Box U_1 \cup \ldots \cup \Box U_n$, $n \geq 1$, where $U_1, \ldots, U_n$ are open in X. Then ν is such that $\nu^* \in [\mathcal{U} > r]$ iff $\nu^*(\Box U_1 \cup \ldots \cup \Box U_n) > r$, iff $\sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu^* \left(\bigcap_{i \in I} \Box U_i \right) > r$, iff $\sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu \left(\bigcap_{i \in I} U_i \right) > r$, i.e., iff $\nu \in [\sum^{\pm} U_1, \ldots, U_n > r]$.

It follows that the soft topology is finer than the weak topology, as $[U > r]$ is the case $n = 1$, $U_1 = U$. The characterization of $\preceq$ also follows easily. $\Box$

Then $\mathbf{V}_{1\ wk}(\mathcal{Q}(X))$ is homeomorphic to $\mathbf{Cd}_1(X)$ with the soft topology, by construction.

So continuous credibilities are models of mixed choice, where the random player goes first, then the demonic non-deterministic player. This is particularly clear on simple credibilities $\sum_{i=1}^n a_i \mathfrak{u}_{Q_i}$: one chooses a set Q_i with probability a_i, then picks non-deterministically from Q_i.

Let us turn to plausibilities. Nicely enough, our duality $\_^\perp$ on games extends our previous duality between $\mathcal{Q}(X)$ and $\mathcal{H}(X)$.

Lemma 6.21. Let X be stably compact. For any $Q \in \mathcal{Q}(X)$, the dual $\mathfrak{u}_Q^\perp$ is $\mathfrak{e}_Q$. For any $F \in \mathcal{H}_{\mathcal{V}}(X)$, the dual $\mathfrak{e}_F^\perp$ is $\mathfrak{u}_F$.

Proof. By definition, for every open $X \setminus Q'$ of X^{d}, $\mathfrak{u}_Q^\perp(X \setminus Q') = 1 - \inf_{U \supseteq Q'} \mathfrak{u}_Q(U)$. If $X \setminus Q'$ intersects Q, then let $x \in Q \setminus Q'$. Since Q' is saturated, and x is not in Q', there must be an open subset U of X containing Q' but not x. In particular, Q is not contained in U, so $\mathfrak{u}_Q(U) = 0$, hence $\mathfrak{u}_Q^\perp(X \setminus Q') = 1$. If $X \setminus Q'$ does not intersect Q, then $Q \subseteq Q'$, so any open U containing Q' contains Q, whence $\mathfrak{u}_Q^\perp(X \setminus Q') = 0$. In any case, $\mathfrak{u}_Q^\perp(X \setminus Q') = \mathfrak{e}_Q(X \setminus Q')$.

The fact that $\mathfrak{e}_F^\perp$ is $\mathfrak{u}_F$ follows by duality: $\mathfrak{e}_F^\perp = \mathfrak{u}_F^{\perp\perp} = \mathfrak{u}_F$. $\Box$

It follows from Lemma 6.13, first part, that the duals $(\sum_{i=1}^n a_i \mathfrak{u}_{Q_i})^\perp$ of simple credibilities are exactly the *simple plausibilities* $\sum_{i=1}^n a_i \mathfrak{e}_{Q_i}$ on the dual space X^{d} (Q_i closed in X^{d}).

Lemma 6.22. Every example game $\mathfrak{e}_A$ is continuous, and $\mathfrak{e}_A = \mathfrak{e}_{cl(A)}$.

For any continuous valuation P on $\mathcal{H}_{\mathcal{V}}(X)$, the game $P_\diamond$ defined by $P_\diamond(U) = P(\diamond U)$ is a continuous plausibility on X. In particular, $\mathfrak{e}_F = \delta_{F\diamond}$ is a continuous plausibility.

The map $P \mapsto P_\diamond$ is monotonic, Scott-continuous, and continuous from $\mathbf{V}_{1\ wk}(\mathcal{H}_{\mathcal{V}}(X))$ to $\mathbf{Pb}_{1\ wk}(X)$.

Proof. For any directed family of opens $(U_i)_{i \in I}$, $\bigcup_{i \in I} U_i$ intersects A if and only if some U_i intersects A, so $\mathfrak{e}_A$ is continuous. For any open U, U intersects $cl(A)$ iff it intersects A, whence $\mathfrak{e}_A = \mathfrak{e}_{cl(A)}$. Let P be a continuous valuation on $\mathcal{H}_{\mathcal{V}}(X)$. P satisfies an equality dual to the inclusion-exclusion principle, which it is tempting to call the *exclusion-inclusion principle*: $P(\bigcap_{i=1}^n \mathcal{U}_i) = \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} P(\bigcup_{i \in I} \mathcal{U}_i)$. Take $\mathcal{U}_i = \diamond U_i$, and notice that $\diamond$ commutes with finite unions, while $\diamond \bigcap_{i=1}^n U_i \subseteq \bigcap_{i=1}^n \diamond U_i$: this shows that $P_\diamond$ is totally concave.

Finally, $\delta_{F\diamond}(U)$ equals 1 iff $F \in \diamond U$, iff $\mathfrak{e}_F(U) = 1$, so $\mathfrak{e}_F = \delta_{F\diamond}$.

The last part of the Lemma is obvious. □

One can relate example games to the Hoare powerdomain, much as unanimity games relate to the Smyth powerdomain (Proposition 6.16, Proposition 6.17):

Proposition 6.23. Let X be a topological space. The map $\mathfrak{e} : F \mapsto \mathfrak{e}_F$ is a continuous order-embedding of $\mathcal{H}(X)$ into $\mathbf{Pb}_1(X)$, and a topological embedding of $\mathcal{H}_{\mathcal{V}}(X)$ into $\mathbf{Pb}_{1\ wk}(X)$.

Proof. That $\mathfrak{e}$ is monotonic is clear. Continuity is easy, too: for every directed family $(F_i)_{i \in I}$ of non-empty closed subsets of X, and every open U, $\mathfrak{e}_{cl(\bigcup_{i \in I} F_i)}(U) = 1$ iff $cl(\bigcup_{i \in I} F_i)$ intersects U, iff $\bigcup_{i \in I} F_i$ intersects U, iff some F_i intersects U, iff $\sup_{i \in I} \mathfrak{e}_{F_i}(U) = 1$. Next, $\mathfrak{e}$ is also continuous from $\mathcal{H}_{\mathcal{V}}(X)$ to $\mathbf{Pb}_{1\ wk}(X)$, since $\mathfrak{e}^{-1}[U > r]$ equals $\diamond U$ if $0 \leq r < 1$, is empty if $r \geq 1$, and the whole of $\mathcal{H}_{\mathcal{V}}(X)$ if $r < 0$. It is an embedding because the image of $\diamond U$ is, say, $[U > 1/2] \cap \mathbf{Pb}_1(X)$, and such sets $\diamond U$ generate the topology of $\mathcal{H}_{\mathcal{V}}(X)$. □

Proposition 6.24. Let X be a topological space. Any normalized continuous plausibility ν such that ν only takes values 0 or 1 is of the form $\mathfrak{e}_F$, $F \in \mathcal{H}_{\mathcal{V}}(X)$. In fact, this already holds of all normalized concave games that take values 0 or 1 only.

Proof. Consider the complement F of the largest open U such that $\nu(U) = 0$. F is non-empty, otherwise $\nu(X) = 0$. For every open set U, if U does not intersect F then $\nu(U) = 0$; if U does, then $\nu(U) \neq 0$, whence $\nu(U) = 1$. So $\nu = \mathfrak{e}_F$. □

Duality allows us to obtain the following characterization of continuous plausibilities as combinations of angelic choice and non-probabilistic choice, almost without effort. We only need one extra lemma.

Lemma 6.25. Let X be stably compact. For every continuous credibility ν on X, for every compact saturated subset Q of X, $\nu^{*\dagger}(\blacksquare Q) = \nu^\dagger(Q)$.

Proof. For every open U containing Q, $\square U$ contains $\blacksquare Q$. It follows that $\nu^{*\dagger}(\blacksquare Q) \leq \inf_{U \supseteq Q} \nu^*(\square U) = \nu^\dagger(Q)$. Conversely, any open $\mathcal{U}$ of $\mathcal{Q}(X)$ is a union of basic opens $\square U$, say $\mathcal{U} = \bigcup_{i \in I} \Box U_i$. If $\mathcal{U}$ contains $\blacksquare Q$, then Q itself is in some $\Box U_i$, that is, $Q \subseteq U_i$. So $\nu^{*\dagger}(\blacksquare Q) = \inf_{\mathcal{U} \supseteq \blacksquare Q} \nu^*(\mathcal{U}) \geq \inf_{U \supseteq Q} \nu^*(\Box U) = \nu^\dagger(Q)$. ☐

Theorem 6.26. Let X be a stably compact space. For every continuous plausibility ν on X there is a unique continuous valuation ν_* on $\mathcal{H}_\mathcal{V}(X)$ such that $\nu(U) = \nu_*(\Diamond U)$ for every open U of X: $\nu_* = \nu^{\perp * \perp}$.

The map $\nu \mapsto \nu_*$ is a one-to-one mapping from $\mathbf{Pb}_1(X)$ to $\mathbf{V}_1(\mathcal{H}_\mathcal{V}(X))$, whose inverse is $P \mapsto P_\Diamond$.

$\mathbf{Pb}_1(X)$ is a continuous dcpo, and with basis given by *simple plausibilities* $\sum_{i=1}^n a_i \mathfrak{e}_{F_i}$ ($a_1, \ldots, a_n \in \mathbb{R}^+$, $F_1, \ldots, F_n \in \mathcal{H}_\mathcal{V}(X)$) that are normalized ($\sum_{i=1}^n a_i = 1$).

Moreover, $(\sum_{i=1}^n a_i \mathfrak{e}_{F_i})_* = \sum_{i=1}^n a_i \delta_{F_i}$.

More precisely, we have the stronger result that any continuous normalized plausibility ν on X is the least upper bound of a family of simple normalized plausibilities $(\nu_i)_{i \in I}$ way-below ν, such that $(\nu_{i*})_{i \in I}$ is also a directed family of simple probabilities way-below ν_* in $\mathbf{V}_1(\mathcal{H}_\mathcal{V}(X))$.

Proof. Observe that for every continuous probability P on $\mathcal{Q}(X^\mathsf{d}) = \mathcal{H}_\mathcal{V}(X)^\mathsf{d}$ (Theorem 3.1), for every open U of X, $P^\perp(\Diamond U) = 1 - P^\dagger(\blacksquare(X \setminus U))$. Indeed, $\Diamond U$ is the complement of $\blacksquare(X \setminus U)$. Take $P = \nu^{\perp *}$: then $P^\perp(\Diamond U) = 1 - \nu^{\perp \dagger}(X \setminus U)$ (by Lemma 6.25) $= \nu^{\perp\perp}(U) = \nu(U)$ (by Lemma 6.10). So $\nu_* = \nu^{\perp * \perp}$ fits the bill. Uniqueness of ν_* follows from uniqueness of $\nu^{\perp *}$ (Theorem 6.18) through duality (Theorem 6.11).

The other claims follow from Lemma 6.21, Lemma 6.13 (first part), and Lemma 6.22. Let indeed ν be any normalized continuous plausibility. Then $\nu^\perp$ is a normalized continuous credibility, and is therefore the least upper bound of some family $(\nu_i^\perp)_{i \in I}$ of duals of simple normalized credibilities way-below ν such that $(\nu_i^{\perp *})_{i \in I}$ is also directed and way-below ν^*, by Theorem 6.18. The claim follows. ☐

Stable compactness is in fact not needed to establish the existence of the one-to-one mapping $\nu \mapsto \nu_*$: in the companion paper (Goubault-Larrecq, 2009, Section 3), we show that core-compactness is the only property required of X to this end. Duality however requires it.

Again $\nu \mapsto \nu_*$ is not continuous, not even monotonic. We let the reader prove the following analogue of Proposition 6.20.

Proposition 6.27 (Dual Soft Topology). Let X be stably compact. Let the *dual soft topology* on $\mathbf{Pb}_1(X)$ be induced from the weak topology on $\mathbf{V}_1(\mathcal{H}_\mathcal{V}(X))$ by $\nu \mapsto \nu_*$, i.e., the topology whose opens are of the form $\{\nu \in \mathbf{Pb}_1(X) \mid \nu_* \in \mathcal{U}\}$, where $\mathcal{U}$ ranges over the open subsets of $\mathbf{V}_{1\ wk}(\mathcal{H}_\mathcal{V}(X))$.

The dual soft topology is finer than the weak topology, in general strictly even on finite spaces. It is generated by the subbasic opens of the form:

$$[\sum_{\pm} U_1, \ldots, U_n > r] \quad = \quad \left\{ \nu \in \mathbf{Pb}_1(X) \mid \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu(\bigcup_{i \in I} U_i) > r \right\}$$

where $n \geq 1$, $r \in \mathbb{R}$, and $U_1, \ldots, U_n$ are open in X. Its specialization ordering $\preceq$ is defined by $\nu \preceq \nu'$ iff, for every $n \geq 1$, for all opens $U_1, \ldots, U_n$,

$$\sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu(\bigcup_{i \in I} U_i) \quad \leq \quad \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu'(\bigcup_{i \in I} U_i)$$

Then $\mathbf{V}_{1\ wk}(\mathcal{H}_{\mathcal{V}}(X))$ is homeomorphic to $\mathbf{Pb}_1(X)$ with the dual soft topology, by construction.

We finally obtain the following natural continuation of Corollary 6.14, as a corollary of Theorem 6.26.

Corollary 6.28. For any stably bicontinuous bicpo X, $\mathbf{Cd}_1(X)$ and $\mathbf{Pb}_1(X)$ are stably bicontinuous bicpos, and $\_^\perp$ is an involutive homeomorphism from $\mathbf{Cd}_1(X)^{\mathsf{op}}$ to $\mathbf{Pb}_1(X^{\mathsf{op}})$ and from $\mathbf{Pb}_1(X)^{\mathsf{op}}$ to $\mathbf{Cd}_1(X^{\mathsf{op}})$.

Every continuous normalized credibility (resp., plausibility) ν on X is both the least upper bound of a directed family of simple normalized credibilities (resp., plausibilities) way-below ν, and the inf of a filtered family of simple normalized credibilities (resp., plausibilities) way-above ν.

7. Mixed Choice II: Previsions

A better model for mixed choice, notably as applied to higher-order programming languages, is given by continuous previsions, and by forks, as argued in (Goubault-Larrecq, 2007b). On stably compact continuous pointed dcpos, the latter are isomorphic to a normalized variant of spaces invented independently by Mislove (Mislove, 2000) and by Tix (Tix, 1999; Tix et al., 2005), as we have shown in (Goubault-Larrecq, 2008a). The latter rests heavily on an extension of the convex-concave duality to previsions. One notes that the asymmetric version of the Kantorovich-Rubinstein that we proved in (Goubault-Larrecq, 2008b) also rests heavily on yet another variant, which occurs as a simplification of the latter when applied to 1-Lipschitz maps.

It is certainly not our goal to reprove these results here. Our purpose is to establish duality theorems for previsions and forks. Some of the basic results appear in (Goubault-Larrecq, 2008a), with only proof sketches or no proof, and the duality results themselves, which we establish below, are new.

For any topological space X, let $\langle X \to \mathbb{R}^+_\sigma \rangle$ be the poset of all bounded continuous maps from X to $\mathbb{R}^+_\sigma$. Recall that we take $\mathbb{R}^+_\sigma$ with the Scott topology, whose non-trivial opens are the open intervals $(t, +\infty)$, $t \in \mathbb{R}^+$, and order $\langle X \to \mathbb{R}^+_\sigma \rangle$ pointwise. Similarly, let $\langle X \to \overline{\mathbb{R}^+} \rangle$ be the dcpo of all continuous maps from X to $\overline{\mathbb{R}^+}$.

Definition 7.1 (Previsions, Forks). A *prevision* F on a topological space X is a monotonic map F from $\langle X \to \mathbb{R}^+_\sigma \rangle$ to $\mathbb{R}^+_\sigma$ such that $F(af) = aF(f)$ for every $a \in \mathbb{R}^+$ (*positive homogeneity*).

F is *continuous* iff it is Scott-continuous.

F is *lower* iff $F(h+h') \geq F(h)+F(h')$ for all h, h', *upper* iff $F(h+h') \leq F(h)+F(h')$ for all h, h', *linear* iff $F(h+h') = F(h)+F(h')$, *normalized* iff $F(a+h) = a+F(h)$ for every function h and every constant $a \in \mathbb{R}^+$, *subnormalized* iff $F(a+h) \leq a+F(h)$ for every h and constant a.

A *fork* is a pair (F^-, F^+) of continuous previsions, where F^- is lower, F^+ is upper, and *Walley's condition*:

$$F^-(h + h') \le F^-(h) + F^+(h') \le F^+(h + h')$$

holds for all h, h'. A fork is normalized, resp. sub-normalized, whenever both F^- and F^+ are.

We let $\mathbf{P}_1(X)$ be the poset of all continuous normalized previsions on X, $\bigtriangledown\mathbf{P}_1(X)$, $\bigtriangleup\mathbf{P}_1(X)$ and $\mathbf{P}_1^{\triangle}(X)$ be the dcpos of those continuous normalized previsions that are lower, resp. upper, resp. linear. $\mathbf{F}_1(X)$ is the space of all normalized forks on X.

We write $\mathbf{P}_{1\ wk}(X)$, $\bigtriangledown\mathbf{P}_{1\ wk}(X)$, $\bigtriangleup\mathbf{P}_{1\ wk}(X)$, $\mathbf{P}_{1\ wk}^{\triangle}(X)$ the corresponding spaces with the *weak topology*, generated by the subbasic opens $[f > r]$, defined as the subset of those previsions F such that $F(f) > r$, $f \in \langle X \to \mathbb{R}^+_\sigma \rangle$, $r \in \mathbb{R}^+$. $\mathbf{F}_{1\ wk}(X)$ is the space $\mathbf{F}_1(X)$ with the *weak topology*, defined as the one induced by the product topology on $\bigtriangledown\mathbf{P}_{1\ wk}(X) \times \bigtriangleup\mathbf{P}_{1\ wk}(X)$.

It is easy to show that $\mathbf{P}_1(X)$, $\bigtriangledown\mathbf{P}_1(X)$, $\bigtriangleup\mathbf{P}_1(X)$, and $\mathbf{P}_1^{\triangle}(X)$ are dcpos, using the fact that addition is Scott-continuous on $\mathbb{R}^+$. Again, the weak topologies are just the topologies of pointwise convergence.

It was shown in (Goubault-Larrecq, 2007b) that, among the continuous normalized previsions, the lower brand was an adequate model of mixed probabilistic and demonically non-deterministic choice, the upper brand was one of mixed probabilistic and angelically non-deterministic choice, while normalized forks were an adequate model of mixed probabilistic and erratically non-deterministic choice. Moreover, they give rise to strong monads, something that is not true of any of the game constructions $\bigtriangledown\mathbf{J}_1$, $\bigtriangleup\mathbf{J}_1$, $\mathbf{Cd}_1$, or $\mathbf{Pb}_1$.

The Scott topology is always finer than the weak topology. When X is a continuous pointed dcpo, the two topologies coincide for continuous linear previsions, i.e., $\mathbf{P}_{1\ wk}^{\triangle}(X) = \mathbf{P}_1^{\triangle}(X)$. This can be shown as follows. First, $\mathbf{V}_{1\ wk}(X) = \mathbf{V}_1(X)$: as we have seen in Section 6, this is a consequence of (Tix, 1995, Satz 4.10) and Edalat's trick. Second, $\mathbf{V}_1(X)$ is isomorphic to $\mathbf{P}_1^{\triangle}(X)$. In one direction, one builds a continuous linear prevision $\alpha_{\mathcal{C}}(\nu)$ for any $\nu \in \mathbf{V}_1(X)$ by integration: $\alpha_{\mathcal{C}}(\nu)(h) = \int_{x \in X} h(x) d\nu$. Conversely, any continuous linear prevision G gives rise to a continuous probability $\gamma_{\mathcal{C}}(G)$: $\gamma_{\mathcal{C}}(G)(U) = G(\chi_U)$, where for every open U, χ_U is the (continuous) map from X to $\mathbb{R}^+_\sigma$ that sends x to 1 if $x \in U$, to 0 otherwise. This is an isomorphism, as noted by (Jung, 2004), who refers to Tix (Tix, 1995, Satz 4.10), who cites Kirch (Kirch, 1993, Satz 8.6). This isomorphism extends to a homeomorphism between $\mathbf{P}_{1\ wk}^{\triangle}(X)$ and the space $\mathbf{V}_1(X)$ equipped with the topology generated by subbasic opens $[f > r] = \{\nu \in \mathbf{V}_1(X) \mid \int_{x \in X} f(x) d\nu > r\}$, $f \in \langle X \to \mathbb{R}^+ \rangle$, $r \in \mathbb{R}^+$. The latter is what Jung calls the weak topology on $\mathbf{V}_1(X)$ (Jung, 2004; Alvarez-Manilla et al., 2004); but the latter is exactly the weak topology in our sense, which Jung calls the product topology (Jung, 2004, Theorem 3.3).

The two maps $\alpha_{\mathcal{C}}$ and $\gamma_{\mathcal{C}}$ extend to general continuous games and previsions. We define instead $\alpha_{\mathcal{C}}(\nu)$ as the Choquet integral $\oint_{x \in X} h(x) d\nu$ (Goubault-Larrecq, 2007b). This integral is extensively studied in (Denneberg, 1994), using the more general setting of integration of measurable maps, and games defined not on a topology but on general algebras. For the sake of completeness, we give a summary of the basic properties of this integral. Since we integrate continuous maps only, integration will commute with sups of directed sets, not just of increasing sequences.

The *Choquet integral* of $h \in \langle X \to \mathbb{R}^+_\sigma \rangle$ along an arbitrary game ν is defined as the ordinary Riemann integral $\int_0^{+\infty} \nu(h^{-1}(t, +\infty))dt$. This is defined, as the map $t \mapsto \nu(h^{-1}(t, +\infty))$ is non-increasing, and maps every $t \geq \sup_{x \in X} h(x)$ to 0; note that every non-increasing map, even non-continuous, is Riemann-integrable on any bounded interval.

Proposition 7.2. For any (continuous) game ν on the topological space X, the map $F = \alpha_\mathcal{C}(\nu)$ is a (continuous) prevision, and $\gamma_\mathcal{C}(\alpha_\mathcal{C}(\nu)) = \nu$. If ν is (sub)normalized, then so is F.

Proof. The Choquet integral is monotonic in the integrated function: if $h \leq h'$, then $h^{-1}(t, +\infty) \subseteq h'^{-1}(t, +\infty)$ for every $t \in \mathbb{R}^+$. Let us show that $\alpha_\mathcal{C}(\nu)$ is continuous as soon as ν is. If $h \in \langle X \to \mathbb{R}^+_\sigma \rangle$ is the sup of a directed family $(h_i)_{i \in I}$ of elements of $\langle X \to \mathbb{R}^+ \rangle$, then $\oint_{x \in X} h(x)d\nu = \int_0^{+\infty} \nu(\bigcup_{i \in I} h_i^{-1}(t, +\infty))dt = \int_0^{+\infty} \sup_{i \in I} \nu(h_i^{-1}(t, +\infty))dt$ (because ν is continuous) $= \sup_{i \in I} \oint_{x \in X} h_i(x)d\nu$. The latter equality uses the fact that Riemann integrals are Scott-continuous in the integrated non-increasing function, as noticed in (Tix, 1995, Lemma 4.2). (This would not hold with other classes of integrated functions. For example, we need to use the Lebesgue integral instead of the Riemann integral just to prove that integration commutes with sups of *countable* non-decreasing sequences of integrable functions.)

Next, $\alpha_\mathcal{C}(\nu)$ is positively homogeneous: for all $a \in \mathbb{R}^+$, $\alpha_\mathcal{C}(\nu)(af) = 0$ if $a = 0$, else $\alpha_\mathcal{C}(\nu)(af) = \int_0^{+\infty} \nu(f^{-1}(t/a, +\infty))dt = a\ \alpha_\mathcal{C}(\nu)(f)$ by change of variables.

Then, $\gamma_\mathcal{C}(\alpha_\mathcal{C}(\nu))(U) = \alpha_\mathcal{C}(\nu)(\chi_U) = \int_0^{+\infty} \nu(\chi_U^{-1}(t, +\infty))dt = \int_0^1 \nu(U)dt + \int_1^{+\infty} 0dt = \nu(U)$, so $\gamma_\mathcal{C}(\alpha_\mathcal{C}(\nu)) = \nu$.

Finally, let ν be (sub)normalized. Then we compute $\alpha_\mathcal{C}(\nu)(f+a) = \oint_{x \in X}(f(x)+a)d\nu = \int_0^{+\infty} \nu(f^{-1}(t-a, +\infty))dt = \int_0^a \nu(X)dt + \int_a^{+\infty} \nu(f^{-1}(t-a, +\infty))dt = a\nu(X) + \alpha_\mathcal{C}(\nu)(f)$. This is less than or equal to $a + \alpha_\mathcal{C}(\nu)(f)$ is ν is subnormalized, and equal to it if ν is normalized. □

Further connections between games and previsions are obtained by appealing to *step functions*.

Definition 7.3. A function $f : X \to \mathbb{R}$ is a *step function* if and only if it is of the form $\sum_{i=0}^n a_i \chi_{U_i}$, where $X = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n$ is a sequence of opens, and $a_0 \in \mathbb{R}$, $a_1, \ldots, a_n \in \mathbb{R}^+$.

It is well-known that any element f of $\langle X \to \mathbb{R}_\sigma \rangle$ is the sup of a directed family of step functions, namely $f_K = a + \frac{1}{2^K} \sum_{k=1}^{\lfloor (b-a)2^K \rfloor} \chi_{f^{-1}(a+\frac{k}{2^K}, +\infty)}$, $K \in \mathbb{N}$, where a is any lower bound for f and b is any upper bound for f.

The following characterizes Choquet integration on step functions that take non-negative values. In this case, a_0 also is in $\mathbb{R}^+$.

Lemma 7.4. For any step function f with non-negative values $\sum_{i=0}^{n} a_i \chi_{U_i}$, where $X = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n$ is a sequence of opens, and $a_0, a_1, \ldots, a_n \in \mathbb{R}^+$:

$$\oint_{x \in X} f(x) d\nu \quad = \quad \sum_{i=0}^{n} a_i \nu(U_i)$$

Proof. Split the Riemann integral $\int_0^{+\infty} \nu(f^{-1}(t, +\infty))dt$ as the sum of integrals from 0 to a_0, from a_0 to $a_0 + a_1$, …, from $a_0 + \ldots + a_{n-1}$ to $a_0 + \ldots + a_{n-1} + a_n$, and from a_n to $+\infty$, then rearrange the sum. □

Since Choquet integration is continuous, this in fact characterizes it completely. Our definition of Choquet integral assumes the integrated function takes only non-negative values. There is a more complicated formula in the case of general real-valued functions. However, we notice that we can define $\oint_{x \in X} f(x)d\nu$, when f is no longer non-negative, as $\widehat{\alpha_{\mathcal{C}}(\nu)}(f)$ (see Definition 7.7 below); then the formula of Lemma 7.4 holds even when $a_0 < 0$.

Lemma 7.5. Let X be a topological space, and $f \in \langle X \to \mathbb{R}^+_\sigma \rangle$.

For every credibility ν on X, if ν^* exists (e.g., when X is well-filtered and locally compact, see Theorem 6.18), then:

$$\oint_{x \in X} f(x) d\nu = \oint_{Q \in \mathcal{Q}(X)} \min_{x \in Q} f(x) d\nu^*.$$

For every plausibility ν on X, if ν_* exists (see Theorem 6.26), then:

$$\oint_{x \in X} f(x) d\nu = \oint_{F \in \mathcal{H}_{\mathcal{V}}(X)} \sup_{x \in F} f(x) d\nu_*.$$

Proof. First, the map $Q \mapsto \min_{x \in Q} f(x)$ is well-defined and continuous. It is well-defined because the image by f of any non-empty compact Q is compact in $\mathbb{R}^+_\sigma$, therefore its saturation is of the form $[r, +\infty)$ for some $r \in \mathbb{R}^+$. Then $r = \min_{x \in Q} f(x)$. It is continuous because the inverse image of $(t, +\infty)$ is $\Box f^{-1}(t, +\infty)$ for all $t \in \mathbb{R}^+$. Now:

$$\oint_{Q \in \mathcal{Q}(X)} \min_{x \in Q} f(x) d\nu^* = \int_0^{+\infty} \nu^*(\Box f^{-1}(t, +\infty))dt = \int_0^{+\infty} \nu(f^{-1}(t, +\infty))dt = \oint_{x \in X} f(x) d\nu$$

using Theorem 6.18 in the middle equation. The other claim is proved similarly, using the fact that the inverse image of $(t, +\infty)$ by the map $F \mapsto \sup_{x \in F} f(x)$ is $\Diamond f^{-1}(t, +\infty)$. □

When the assumptions of Theorem 6.18 are satisfied, this makes it clear how continuous credibilities encode probabilistic choice first, of some $Q \in \mathcal{Q}(X)$, drawn at random along the valuation ν^*, followed by demonic non-deterministic choice. *Demonic* takes a clear meaning here: the non-deterministic player tries to *minimize* the gain $f(x)$ by computing $\min_{x \in Q} f(x)$ for each choice of Q. Similary, under the assumptions of Theorem 6.26, where ν is a continuous plausibility, an *angelic* player tries to maximize the gain.

The formulae above are probably a bit clearer if one realizes that integrating f along the valuation δ_x yields $f(x)$. When ν is a simple credibility $\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$, ν^* exists, without any assumption on the ambient space X, and equals $\sum_{i=1}^n a_i \delta_{Q_i}$, so $\alpha_{\mathcal{C}}(\nu)$ maps each $h \in \langle X \to \mathbb{R}^+_\sigma \rangle$ to $\sum_{i=1}^n a_i \min_{x \in Q_i} h(x)$: this is the average of all possible least gains $\min_{x \in Q_i}$, when Q_i is chosen with probability a_i—a *mean of mins* in the words of (Gilboa and Schmeidler, 1994, Theorem 4.3). Similarly, if $\nu = \sum_{i=1}^n a_i \mathfrak{e}_{F_i}$ is a simple plausibility, then $\alpha_{\mathcal{C}}(\nu)(h) = \sum_{i=1}^n a_i \sup_{x \in F_i} h(x)$.

The demonic variant of Shapley's Theorem (Goubault-Larrecq, 2007b, Theorem 4) states that any continuous normalized lower prevision F has a non-empty *continuous heart* $CCoeur_1(F) = \{G \in \mathbf{P}_1^{\triangle}(X) \mid F \leq G\}$, provided X is stably compact. This is, up to isomorphism, the *core* of F, namely the set of all probabilities p such that $\oint_{x \in X} h(x) dp$ is greater than or equal to $F(h)$ for all $h \in \langle X \to \mathbb{R}^+_\sigma \rangle$. The demonic variant of Rosenmuller's Theorem (Goubault-Larrecq, 2007b, Theorem 5) states that, if we fix $h \in \langle X \to \mathbb{R}^+_\sigma \rangle$, then there is even one such p such that $\oint_{x \in X} h(x) dp$ *equals* $F(h)$. In particular, for all h, $F(h) = \min_{G \in CCoeur_1(F)} G(h) = \min_{p \text{ in the core of } F} \oint_{x \in X} h(x) dp$, showing that continuous normalized lower previsions operate a mixed choice. This time, the demonic player plays first, and finds a probability that minimizes the average gain of the random player—a *min of means*. This is the starting point of (Goubault-Larrecq, 2008a), where we show that the space of continuous normalized lower previsions is isomorphic to the subspace of so-called strongly convex elements of $\mathcal{Q}(\mathbf{V}_{1\ wk}(X))$. This is the demonic case. The angelic and erratic cases are also dealt with, using convex-concave duality; this culminates in theorems showing that the prevision and fork models are isomorphic to the Mislove-Tix-Keimel-Plotkin models on stably compact, continuous, pointed dcpos. We won't go as far: we shall have enough work to study convex-concave duality itself.

We finish our study of the relations between games and previsions by the following.

Proposition 7.6. If ν is a continuous convex game, then $\alpha_{\mathcal{C}}(\nu)$ is a continuous lower prevision. If ν is a continuous concave game, then $\alpha_{\mathcal{C}}(\nu)$ is a continuous upper prevision. If ν is a continuous valuation, then $\alpha_{\mathcal{C}}(\nu)$ is a continuous linear prevision.

Proof. The main ingredient is to notice that, for any opens U, V of X, $\chi_{U \cup V} + \chi_{U \cap V} = \chi_U + \chi_V$. Let ν be a convex game. Let g be a step function of the form $\epsilon \sum_{i=1}^n \chi_{U_i}$, $U_1 \supseteq \ldots \supseteq U_n$, $\epsilon > 0$. By convention let $U_i = X$ for all $i \leq 0$, and $U_i = \emptyset$ for all $i > n$. We first observe that for any open V, $g + \epsilon \chi_V = \epsilon \sum_{i=1}^{n+1} \chi_{W_i}$ where $W_i = (U_{i-1} \cap V) \cup U_i$ for all $i \in \mathbb{Z}$ forms a non-increasing family of opens. Indeed, $\chi(W_i) = \chi_{U_{i-1} \cap V} + \chi_{U_i} - \chi_{U_i \cap V}$ (since $U_i \subseteq U_{i-1}$), then cancel the first and last terms in pairs in the sum. So $\oint_{x \in X} (g(x) + \epsilon \chi_V(x)) d\nu = \epsilon \sum_{i=1}^n \nu(W_i) \geq \epsilon \sum_{i=1}^n [\nu(U_{i-1} \cap V) + \nu(U_i) - \nu(U_i \cap V)] = \epsilon \nu(V) + \oint_{x \in X} g(x) d\nu$, where the $\geq$ sign follows from the convexity of ν. By induction on m, it follows that whenever $g'(x)$ is another step function of the form $\epsilon \sum_{j=1}^m \chi_{V_j}$, $V_1 \supseteq \ldots \supseteq V_m$, $\oint_{x \in X} (g(x) + g'(x)) d\nu \geq \oint_{x \in X} g'(x) d\nu + \oint_{x \in X} g(x) d\nu$. For all $f, f' \in \langle X \to \mathbb{R}^+_\sigma \rangle$, f is a sup of a directed family $(f_K)_{K \in \mathbb{N}}$ (see comment after Definition 7.3, with $a = 0$, $\epsilon = 1/2^K$), and similarly for f'. Taking $g = f_K$, $g' = f'_K$, we have just shown that $\alpha_{\mathcal{C}}(\nu)(f_K + f'_K) \geq \alpha_{\mathcal{C}}(\nu)(f_K) + \alpha_{\mathcal{C}}(\nu)(f'_K)$. By Scott-continuity (Proposition 7.2), $\alpha_{\mathcal{C}}(\nu)(f + f') \geq \alpha_{\mathcal{C}}(\nu)(f) + \alpha_{\mathcal{C}}(\nu)(f')$, so $\alpha_{\mathcal{C}}(\nu)$ is lower. The other claims are proved similarly. In fact, $\alpha_{\mathcal{C}}(\nu)$ is lower, resp. upper, resp. linear when ν is convex, resp. concave, resp. modular, without assuming that ν is continuous (Goubault-Larrecq, 2007, Section 4.3); this just requires a bit more care. □

One should note that the converse of Proposition 7.6 does not hold: there are continuous (lower, upper) previsions that do not arise from continuous (convex, concave) games. E.g., on $X = \{1, 2\}$ with the discrete topology, any convex game is of the form $a_1\delta_1 + a_2\delta_2 + a_{12}\mathfrak{u}_X$, for some reals $a_1, a_2 \geq 0$ and a_{12} (not necessarily non-negative) such that $a_{12} \geq -a_1 - a_2$ (Gilboa and Schmeidler, 1994). Then $\alpha_{\mathcal{C}}(\nu)(h) = a_1 h(1) + a_2 h(2) + a_{12} \min(h(1), h(2))$. This is a piecewise linear function of $h(1)$, $h(2)$. However, there are other, non-piecewise linear functions that arise as continuous convex previsions, e.g., the entropy functional $F(h) = \frac{h(1)}{h(1)+h(2)} \log \frac{h(1)}{h(1)+h(2)} + \frac{h(2)}{h(1)+h(2)} \log \frac{h(2)}{h(1)+h(2)}$. In fact, all continuous previsions arising from games through $\alpha_{\mathcal{C}}$ are *collinear*, meaning that $\alpha_{\mathcal{C}}(\nu)(h + h') = \alpha_{\mathcal{C}}(\nu)(h + h')$ for all $h, h' \in \langle X \to \mathbb{R}^+\rangle$ that are *comonotonic*, i.e., such that there are no two points $x, x' \in X$ such that $h(x) < h(x')$ and $h'(x') < h'(x)$. One can show that $\alpha_{\mathcal{C}}$ then defines an isomorphism between spaces of continuous (convex, concave) games and of continuous *collinear* (lower, upper) previsions (Goubault-Larrecq, 2007b, Theorem 1). Again, this would lead us too far astray.

We turn to the core of this section: the *convex-concave duality* on spaces of previsions. This takes a very simple form, if we ignore technical details: the dual of F should be $F^\perp = \lambda h \in \langle X \to \mathbb{R}^+_\sigma\rangle \cdot -F(-h)$. If F is lower, then $F^\perp$ will be upper, and conversely, moreover $F^{\perp\perp} = F$.

Unfortunately, $F(-h)$ is in general ill-defined: First, $-h$ does not take its values in $\mathbb{R}^+_\sigma$ (easy to repair, see below); second, $-h$ is far from being continuous from X to $\mathbb{R}^+_\sigma$: the inverse image of the open $(t, +\infty)$ by $-h$ is $h^{-1}(-\infty, -t)$, of which we know nothing. We shall therefore approximate continuous maps h by perfect maps g (Definition 4.6), noticing that whenever $g : X \to \mathbb{R}_\sigma$ is perfect, then so is $-g : X^{\mathsf{d}} \to \mathbb{R}_\sigma$, as soon as X is stably compact.

To correct the first problem, extend any normalized prevision F on X to a new functional $\widehat{F}$. Let $\langle X \to \mathbb{R}_\sigma\rangle$ be the space of all bounded continuous maps from X to $\mathbb{R}$, with the Scott topology of the pointwise ordering.

Definition 7.7. Let X be a stably compact space. For any normalized prevision F on X, define $\widehat{F} : \langle X \to \mathbb{R}_\sigma\rangle \to \mathbb{R}_\sigma$ by:

$$\widehat{F}(h) = F(h + a) - a,$$

for any $a \geq -\inf_{x \in X} h(x)$.

This is independent of a, because F is normalized, so $\widehat{F}$ is well-defined.

Lemma 7.8. For any normalized prevision F, $\widehat{F}$ is *monotonic*, i.e., for any two maps $h, h' \in \langle X \to \mathbb{R}_\sigma\rangle$, if $h \leq h'$ then $\widehat{F}(h) \leq \widehat{F}(h')$; $\widehat{F}$ is *positively homogeneous*, i.e., for all $r \in \mathbb{R}^+_\sigma$ and $h \in \langle X \to \mathbb{R}_\sigma\rangle$, $\widehat{F}(rh) = r\widehat{F}(h)$; $\widehat{F}$ is *normalized*, i.e., $a + \widehat{F}(h) = \widehat{F}(a + h)$ for all $a \in \mathbb{R}$; finally, $\widehat{F}$ is lower, resp. upper, resp. linear whenever F is.

Proof. Monotonicity is clear. Positive homogeneity: assume $\inf_{x \in X} h(x) < 0$, otherwise the claim is clear; if $r = 0$, then $\widehat{F}(rh) = F(0) - 0 = 0$, else pick r larger than

$-\inf_{x \in X} h(x)$, then $\widehat{F}(rh) = F(rh + ra) - ra = r(F(h + a) - a) = r\widehat{F}(h)$. Normalization: for all $a \in \mathbb{R}$, let $a' \geq -\inf_{x \in X}(a + h)$, then $\widehat{F}(a + h) = F(h + a + a') - a' = F(h + (a + a')) - (a + a') + a = \widehat{F}(h) + a$. The last claim is clear. □

Solving the second problem will be done through the approximation of continuous maps by perfect maps, as we have said above. It has been observed several times that, when X is stably compact, every continuous map h from X to $\mathbb{R}_\sigma$ is the sup of a directed family of perfect maps g below h (Edwards, 1978); see also (Alvarez-Manilla et al., 2004, Lemma 17), or (Lawson, 1991, Theorem 13). We give a proof in Proposition 7.11 below.

Then we will define $F^\perp(g)$ as $-\widehat{F}(-g)$ for all (bounded) perfect maps $g : X^{\mathsf{d}} \to \mathbb{R}^+_\sigma$, and extend by continuity to all bounded continuous maps from X^{d} to $\mathbb{R}^+$.

We now observe that we can even approximate h by perfect maps $g \ll h$. The following lemma is due to one of the anonymous referees:

Lemma 7.9. Let X be a stably compact space, and g a perfect map from X to $\mathbb{R}^+_\sigma$. For every $\epsilon > 0$, the map $g_\epsilon : x \mapsto \max(g(x) - \epsilon, 0)$ is perfect and $g_\epsilon \ll g$ in $\langle X \to \mathbb{R}^+_\sigma \rangle$.

Proof. First, g_ϵ is perfect, as a composition of perfect maps.

Second, let $(f_i)_{i \in I}$ be a directed family in $\langle X \to \mathbb{R}^+_\sigma \rangle$ such that $\sup_{i \in I} f_i$ exists and $g \leq \sup_{i \in I} f_i$. For every $x \in X$ such that $g(x) > \epsilon$, let i_x be an element of I such that $f_{i_x}(x) > g(x) - \epsilon/2$. Then consider $U_x = \{z \in X \mid f_{i_x}(z) > g(z) - \epsilon/2\}$. Note that $U_x = \{z \in X \mid \exists t \in \mathbb{R} \cdot f_{i_x}(z) > t \text{ and } t > g(z) - \epsilon/2\} = \bigcup_{t \in \mathbb{R}} f_{i_x}^{-1}(t, +\infty) \setminus g^{-1}[t + \epsilon/2, +\infty)$ is patch-open, since f_{i_x} is continuous and g is perfect. Also, U_x contains x.

Let $Q = g^{-1}[\epsilon, +\infty)$: this is compact since g is perfect, and therefore also patch-closed, hence patch-compact. Since Q is contained in $\bigcup_{\substack{x \in X \\ g(x) > \epsilon/2}} U_x$, there is a finite set A of elements $x \in X$ with $g(x) > \epsilon/2$ such that $Q \subseteq \bigcup_{x \in A} U_x$. That is, for every $z \in X$ such that $g(z) \geq \epsilon$ (i.e., $z \in Q$), there is an $x \in A$ such that $f_{i_x}(z) > g(z) - \epsilon/2$. Since $(f_i)_{i \in I}$ is directed, pick an $i \in I$ such that $f_{i_x} \leq f_i$ for all $x \in A$: for every $z \in X$ such that $g(z) \geq \epsilon$, we obtain that $f_i(z) > g(z) - \epsilon/2$. It follows that $f_i(z) \geq g_\epsilon(z)$ for all $z \in X$. So $g_\epsilon \ll g$. □

The Urysohn-Nachbin Lemma (Gierz et al., 2003, Exercise VI-1.16) states that in any monotone normal pospace, for every upward-closed closed subset Q, for every downward-closed closed subset F such that $Q \cap F = \emptyset$, there is a continuous order-preserving map $f : X \to [0, 1]$ that is identically 0 on F and identically 1 on Q. This certainly applies to compact pospaces, by Proposition VI.1.8 of op.cit., so:

Lemma 7.10. Let X be stably compact. For every compact saturated subset Q and for every closed subset F such that $Q \cap F = \emptyset$, there is a perfect map $g : X \to [0, 1]_\sigma$ that is identically 0 on F and identically 1 on Q.

Proposition 7.11. Let X be stably compact. Then $\langle X \to \mathbb{R}^+_\sigma \rangle$ is a continuous poset, with a basis of perfect maps; $\langle X \to [0, 1]_\sigma \rangle$ is a continuous dcpo, with a basis of perfect maps.

Proof. Since X is locally compact, for each open subset U of X, U is the union of the directed family of all interiors $int(Q)$ of compact saturated subsets Q of U. Consider the family of all perfect maps $g : X \to [0,1]_\sigma$ that are identically 0 outside of U. This is a directed family, and Lemma 7.10 entails that for each compact saturated subset Q of U, there is one element of this family that is identically 1 on Q. It follows that the sup of this family is χ_U.

In particular, any step function $1/2^K \sum_{k=1}^N \chi_{U_k}$ is also the sup of a directed family of perfect maps, namely those of the form $1/2^K \sum_{k=1}^N g_k$ where for each k, g_k is perfect, with values in $[0,1]$, and identically 0 outside of U_k. Given any directed family D of perfect maps g whose sup is $1/2^K \sum_{k=1}^N \chi_{U_k}$, $D(\epsilon) = \{g_\epsilon \mid g \in D\}$ is also a directed family of perfect maps, and these are way-below $1/2^K \sum_{k=1}^N \chi_{U_k}$.

Any bounded continuous map f from X to $\mathbb{R}^+$ is the sup of a directed family of step functions $(f_i)_{i \in I}$. It is standard that if each f_i is itself the sup of a directed family of elements $(g_{ij})_{j \in J_i}$ way-below f_i (here, consisting of perfect maps), then f is itself the sup of the family $(g_{ij})_{\substack{i \in I \\ j \in J_i}}$, and the latter is directed. □

We can now define $F^\perp$.

Definition 7.12 ($F^\perp$). Let X be a stably compact space, F a normalized prevision on X. The *dual* $F^\perp$ of F is the unique continuous map from $\langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma \rangle$ to $\mathbb{R}^+_\sigma$ such that:

$$F^\perp(h) = -\widehat{F}(-h)$$

for all perfect maps h from X^{d} to $\mathbb{R}^+$.

This is given by Scott's formula: $F^\perp(h) = a + \sup_{g \ll^{\mathsf{d}} h} -F(a-g)$, where g ranges over perfect maps from X^{d} to $\mathbb{R}^+_\sigma$, and $a \geq \sup_{x \in X} h(x)$. We use the notation $\ll^{\mathsf{d}}$ to make it clear that we are using the way-below relation on $\langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma \rangle$, not on $\langle X \to \mathbb{R}^+_\sigma \rangle$.

Proposition 7.13. Let X be stably compact, F a normalized prevision on X. Then $F^\perp$ is a continuous, normalized prevision on X^{d}. If F is continuous, then $F^{\perp\perp} = F$.

Proof. By Lemma 5.7, $F^\perp$ is continuous (and monotonic). We must show that it is positively homogeneous. First, $F^\perp(0) = -F(-0) = 0$ since 0 is perfect. For every $\alpha > 0$, by Lemma 7.8 $F^\perp(\alpha h) = \alpha F^\perp(h)$ whenever h is perfect, so the map $h \mapsto \frac{1}{\alpha} F^\perp(\alpha h)$ is a (necessarily continuous) map from $\langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma \rangle$ to $\mathbb{R}^+_\sigma$ that coincides with $F^\perp$ on perfect maps. By uniqueness, they are equal, i.e., $F^\perp(\alpha h) = \alpha F^\perp(h)$ for all continuous h. We show that $F^\perp$ is normalized, i.e., that $F^\perp(a+h) = a + F^\perp(h)$ for all h, in a similar way, considering the map $h \mapsto F^\perp(a+h) - a$. That $F^{\perp\perp} = F$ when F is continuous follows from the fact that $F^{\perp\perp}$ and F coincide on perfect maps. □

Our next step is to show that $F^\perp$ is lower whenever F is upper and conversely. This requires us to show that addition on $\langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma \rangle$ preserves and reflects $\ll^{\mathsf{d}}$ (recall Definition 5.8). The cone $\langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma \rangle$ will then be *additive*, a notion defined for d-cones in (Keimel, 2006, Definition before Lemma 4.2).

We deal with $\langle X \to \mathbb{R}^+_\sigma \rangle$ instead of $\langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma \rangle$, to avoid carrying d exponents. That addition reflects $\ll$ follows from the fact that addition is Scott-continuous:

Lemma 7.14. Let X be stably compact, and f, g two bounded continuous functions from X to $\mathbb{R}^+_\sigma$. If $h \ll f + g$, then there are perfect maps f', g' from X to $\mathbb{R}^+_\sigma$ such that have $h \leq f' + g'$, $f' \ll f$ and $g' \ll g$.

Proof. Let B_f the set of all perfect maps f' such that $f' \ll f$, B_g that of all perfect maps g' such that $g' \ll g$. By Proposition 7.11, $f = \sup_{f' \in B_f} f'$, $g = \sup_{g' \in B_g} g'$. Since addition is Scott-continuous, $f + g = \sup_{f' \in B_f, g' \in B_g} f' + g'$. Moreover, B_f and B_g are directed, so $B_f \times B_g$ is, too. Since $h \ll f + g$, there are $f' \in B_f$ and $g' \in B_g$ such that $h \leq f' + g'$. ☐

Addition also preserves $\ll$.

Lemma 7.15. Let X be stably compact. For all bounded continuous functions f, f', h, h' from X to $\mathbb{R}^+_\sigma$, if $f \ll h$ and $f' \ll h'$, then $f + f' \ll h + h'$.

Proof. By Proposition 7.11, we can appeal to interpolation: there are perfect maps g and g' such that $f \ll g \ll h$ and $f' \ll g' \ll h'$. Since $(g_\epsilon)_{\epsilon > 0}$ is a directed family whose sup is g, and $f \ll g$, for every small enough $\epsilon > 0$, $f \leq g_\epsilon$. Similarly, we may require that $\epsilon > 0$ be small enough that $f' \leq g'_\epsilon$. Then $f + f' \leq g_\epsilon + g'_\epsilon \leq (g + g')_\epsilon$, so that $f + f' \ll g + g'$, by Lemma 7.9. Therefore $f + f' \ll h + h'$. ☐

Proposition 7.16. Let X be stably compact, F a normalized prevision on X. If F is lower, then $F^\perp$ is upper. If F is upper, then $F^\perp$ is lower.

Proof. Let F be lower, i.e., super-additive. Then for each $h, h' \in \langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma \rangle$, letting $a \geq \sup_{x \in X} h(x)$ and $a' \geq \sup_{x \in X} h'(x)$:

$$\begin{aligned}
F^\perp(h) + F^\perp(h') &= a + \sup_{g \text{ perfect} \ll^{\mathsf{d}} h} (-F(a - g)) + a' + \sup_{g' \text{ perfect} \ll^{\mathsf{d}} h'} (-F(a' - g')) \\
&= a + a' + \sup_{\substack{g \text{ perfect} \ll^{\mathsf{d}} h \\ g' \text{ perfect} \ll^{\mathsf{d}} h'}} (-F(a - g) - F(a' - g')) \\
&\geq a + a' + \sup_{\substack{g \text{ perfect} \ll^{\mathsf{d}} h \\ g' \text{ perfect} \ll^{\mathsf{d}} h'}} (-F(a + a' - g - g')) \\
&= a + a' + \sup_{g'' \text{ perfect} \ll^{\mathsf{d}} h + h'} (-F(a + a' - g'')) = F^\perp(h + h')
\end{aligned}$$

where in the last line we have used that addition preserves (Lemma 7.15) and reflects (Lemma 7.14) $\ll^{\mathsf{d}}$. Similarly, $F^\perp$ is lower whenever F is upper. ☐

Recall that we say that $\_^\perp$ is involutive iff $\_^{\perp\perp}$ is the identity map. To sum up:

Theorem 7.17 (Duality, Previsions, Order-Theoretic). Let X be a stably compact space. For every normalized prevision F on X, $F^\perp$ is a normalized prevision on X^{d}. Moreover: (1) $F^\perp$ is continuous; (2) if F is lower, then $F^\perp$ is upper; (3) if F is upper, then $F^\perp$ is lower; (4) if F is linear, then so is $F^\perp$; (5) if F is continuous, then $F^{\perp\perp} = F$; (6) if $F \leq F'$ then $F'^\perp \leq F^\perp$.

It follows that $F \mapsto F^\perp$ is an involutive order-isomorphism:

— from $\mathbf{P}_1(X)^{\mathsf{op}}$ to $\mathbf{P}_1(X^{\mathsf{d}})$;
— from $\bigtriangledown \mathbf{P}_1(X)^{\mathsf{op}}$ to $\bigtriangleup \mathbf{P}_1(X^{\mathsf{d}})$;

— from $\bigtriangleup \mathbf{P}_1(X)^{\mathsf{op}}$ to $\bigtriangledown \mathbf{P}_1(X^{\mathsf{d}})$;
— from $\mathbf{P}_1^{\triangle}(X)^{\mathsf{op}}$ to $\mathbf{P}_1^{\triangle}(X^{\mathsf{d}})$.

Proof. (1) and (5) are by Proposition 7.13. (2) and (3) are by Proposition 7.16. (4) is a trivial consequence of (2) and (3). (6) is obvious. The rest is trivial. ☐

As in Corollary 6.12, this implies corresponding order-isomorphisms from $\mathbf{P}_1(X)^{\mathsf{op}}$ to $\mathbf{P}_1(X^{\mathsf{op}})$, from $\bigtriangledown \mathbf{P}_1(X)^{\mathsf{op}}$ to $\bigtriangleup \mathbf{P}_1(X^{\mathsf{op}})$, from $\bigtriangleup \mathbf{P}_1(X)^{\mathsf{op}}$ to $\bigtriangledown \mathbf{P}_1(X^{\mathsf{op}})$, and finally from $\mathbf{P}_1^{\triangle}(X)^{\mathsf{op}}$ to $\mathbf{P}_1^{\triangle}(X^{\mathsf{op}})$, when X is a stably bicontinuous bicpo. The resulting spaces are again stably bicontinuous bicpos. This is a consequence of the above results and (Goubault-Larrecq, 2008a, Theorem 3, Theorem 6), whose proofs depend on Theorem 7.17 above, but require too much extra machinery to be included here.

Corollary 7.18 (Duality, Forks, Order-Theoretic). Let X be stably compact. For every normalized fork $F = (F^-, F^+)$, define $F^\perp$ as $(F^{+\perp}, F^{-\perp})$. Then $F^\perp$ is a normalized fork on X^{d}, and $\_^\perp$ defines an involutive order-isomorphism from $\mathbf{F}_1(X)^{\mathsf{op}}$ to $\mathbf{F}_1(X^{\mathsf{d}})$.

Proof. The only thing to check is that $F^\perp$ satisfies Walley's condition. For every $h, h' \in \langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma\rangle$, with $a \geq \sup_{x\in X} h(x)$ and $a' \geq \sup_{x \in X} h'(x)$,

$$\begin{aligned} F^{-\perp}(h) + F^{+\perp}(h') &= a + \sup_{g \text{ perfect} \ll^{\mathsf{d}} h} (-F^-(a-g)) + a' + \sup_{g' \text{ perfect} \ll^{\mathsf{d}} h'} (-F^+(a'-g')) \\ &= a + a' + \sup_{\substack{g \text{ perfect} \ll^{\mathsf{d}} h \\ g' \text{ perfect} \ll^{\mathsf{d}} h'}} (-F^-(a-g) - F^+(a'-g')) \\ &\geq a + a' + \sup_{\substack{g \text{ perfect} \ll^{\mathsf{d}} h \\ g' \text{ perfect} \ll^{\mathsf{d}} h'}} (-F^+(a+a'-g-g')) \end{aligned}$$

using the right-hand side of Walley's condition on (F^-, F^+). Since addition preserves and reflects $\ll^{\mathsf{d}}$ (Lemma 7.15, Lemma 7.14), the latter equals $F^{+\perp}(h+h')$. This yields the left-hand side of Walley's condition on $(F^{+\perp}, F^{-\perp})$. The other side is similar. ☐

Again, if X is a stably bicontinuous bicpo, this induces an involutive order-isomorphism from $\mathbf{F}_1(X)^{\mathsf{op}}$ to $\mathbf{F}_1(X^{\mathsf{op}})$. Also, $\mathbf{F}_1(X)$ is a stably bicontinuous bicpo, as a consequence of (Goubault-Larrecq, 2008a, Section 6).

The order-theoretic duality above extends to a duality on spaces with their weak topologies, as before. We observe that $\ll$ and $\ll^{\mathsf{d}}$ are nicely related. Up to some details, $g \ll g'$ iff $-g' \ll^{\mathsf{d}} -g$:

Lemma 7.19. Let X be stably compact, and write $\ll$ the way-below relation on $\langle X \to \mathbb{R}^+_\sigma\rangle$, $\ll^{\mathsf{d}}$ the way-below relation on $\langle X^{\mathsf{d}} \to \mathbb{R}^+_\sigma\rangle$.

For all perfect maps g, g' from X to $\mathbb{R}^+_\sigma$, such that $\inf_{x\in X} g'(x) > 0$, for every constant a such that $a \geq \sup_{x\in X} g'(x)$ and $a > \sup_{x \in X} g(x)$, $g \ll g'$ iff $a - g' \ll^{\mathsf{d}} a - g$.

Proof. Assume $g \ll g'$. Since g' is the sup of the directed family of maps $(g'_\epsilon)_{\epsilon > 0}$, $g \leq g'_\epsilon$ for some $\epsilon > 0$. Pick also ϵ small enough that $\epsilon < \inf_{x\in X} g'(x)$, which is possible since $\inf_{x \in X} g'(x) > 0$. Then $g(x) \leq g'(x) - \epsilon$ for all $x \in X$, so $a - g'(x) \leq a - g(x) - \epsilon$ for all $x \in X$. So $a - g' \leq (a-g)_\epsilon$, which implies $a - g' \ll^{\mathsf{d}} a - g$ by Lemma 7.9.

The converse direction is similar. □

We shall also need the following lemma.

Lemma 7.20. Let X be compact. For every bounded continuous map f from X to $\mathbb{R}^+_\sigma$, and every $a > 0$, $f \ll a\chi_X$ iff $f \leq a'\chi_X$ for some $a' \in \mathbb{R}^+$ with $a' < a$.

Proof. Assume $f \ll a\chi_X$. Since $a\chi_X$ is the sup of the directed family of all $a'\chi_X$, $a' < a$, $f \leq a'\chi_X$ for some a' with $0 \leq a' < a$. Conversely, assume $f \leq a'\chi_X$ with $a' < a$. Let $(f_i)_{i \in I}$ be any directed family of bounded continuous maps having a sup above $a\chi_X$. Then $(f_i^{-1}(a', +\infty))_{i \in I}$ is a directed family of opens whose union contains X: for every $x \in X$, $\sup_{i \in I} f_i(x) \geq a > a'$, so $x \in f_i^{-1}(a', +\infty)$ for some $i \in I$. Since X is compact, X is contained in $f_i^{-1}(a', +\infty)$ for some $i \in I$. So, for every $x \in X$, $f_i(x) > a' \geq f(x)$. Hence $f \leq f_i$. □

The following theorem is proved using arguments that should be familiar by now, so we go faster.

Theorem 7.21 (Duality, Previsions, Topological Version). Let X be stably compact, then $\mathbf{P}_{1\ wk}(X)$, $\bigtriangledown \mathbf{P}_{1\ wk}(X)$, $\bigtriangleup \mathbf{P}_{1\ wk}(X)$ and $\mathbf{P}^{\triangle}_{1\ wk}(X)$ are stably compact, and $F \mapsto F^\perp$ is an involutive homeomorphism:

— from $\mathbf{P}_{1\ wk}(X)^{\mathsf{d}}$ to $\mathbf{P}_{1\ wk}(X^{\mathsf{d}})$;
— from $\bigtriangledown \mathbf{P}_{1\ wk}(X)^{\mathsf{d}}$ to $\bigtriangleup \mathbf{P}_{1\ wk}(X^{\mathsf{d}})$;
— from $\bigtriangleup \mathbf{P}_{1\ wk}(X)^{\mathsf{d}}$ to $\bigtriangledown \mathbf{P}_{1\ wk}(X^{\mathsf{d}})$;
— from $\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}$ to $\mathbf{P}^{\triangle}_{1\ wk}(X^{\mathsf{d}})$.

Proof. For every perfect map h from X^{d} to $\mathbb{R}^+_\sigma$, and $r \in \mathbb{R}$, let $P_{h \geq r}$ be the property of normalized previsions defined to hold of F iff $\widehat{F}(-h) \geq r$.

Consider $Z = \prod_{\substack{f \in \langle X \to \mathbb{R}^+_\sigma \rangle \\ \sup_{x \in X} f(x) = 1}} [0,1]_\sigma$; Z is stably compact. For each $z \in Z$, write z_f the f component of z. For any conjunction P of properties of previsions among "lower", "upper", "linear", "normalized", $P_{h \geq r}$ for any h and r as above, let $\mathbf{P}^P(X)$ be the space of continuous subnormalized previsions satisfying P, and $\mathsf{P}^P(X)$ the space of all subnormalized previsions satisfying P. There is an obvious map $e : \mathsf{P}^P(X) \to Z$ that sends F to the family of all $F(f)$, $f \in \langle X \to \mathbb{R}^+_\sigma \rangle$, $\sup_{x \in X} f(x) = 1$. Conversely, for any family $z = (z_f)_{\substack{f \in \langle X \to \mathbb{R}^+_\sigma \rangle \\ \sup_{x \in X} f(x) = 1}}$ of elements of $[0,1]_\sigma$, one defines a positively homogeneous functional $m(z)$ from $\langle X \to \mathbb{R}^+_\sigma \rangle$ to $\mathbb{R}^+_\sigma$ by: $m(z)(0) = 0$, $m(z)(f) = a z_{f/a}$ when $a = \sup_{x \in X} f(x) > 0$.

The subspace Z^P of Z of those z such that $m(z)$ is a subnormalized prevision satisfying P is definable by a system of patch-continuous inequalities, with $A = [0,1]_\sigma$. We write the following inequalities, depending on P:

— Monotonic: we would like to write the inequality $a \times \_(f/a) \stackrel{.}{\leq} b \times \_(g/b)$, for all non-identically zero maps $f, g \in \langle X \to \mathbb{R}^+_\sigma \rangle$ such that $f \leq g$, where $a = \sup_{x \in X} f(x)$, $b = \sup_{x \in X} g(x)$. This states that $m(z)(f) \leq m(z)(g)$ for all $f \leq g$, when $f \neq 0$; if $f = 0$, $m(z)(f) \leq m(z)(g)$ always holds. However, the two sides of the inequality may fail to be in $[0,1]_\sigma$, so we divide the two sides by b—the *normalization factor*—, and write $\frac{a}{b} \times \_(f/a) \dot{\leq} \_(g/b)$.

— Subnormalized: $(a+b) \times \_((a+f)/(a+b)) \dot{\leq} a + b \times \_(f/b)$ for every non-identically zero $f \in \langle X \to \mathbb{R}^+_\sigma \rangle$ and every $a > 0$, where $b = \sup_{x \in X} f(x)$; this states that $m(z)(a+f) \leq a + m(z)(f)$ for every $f \neq 0$; when $f = 0$, $m(z)(a) = a \times m(z)(a/a) \leq a$ always holds, and when $a = 0$, $m(z)(f) \leq m(z)(f)$ is trivial. We again need to divide by a normalization factor, here $a + b$.

— Normalized: turn the above into an equality.

— Lower: $a \times \_(f/a) + b \times \_(g/b) \dot{\leq} c \times \_((f+g)/c)$ for all non-identically zero maps $f, g \in \langle X \to \mathbb{R}^+_\sigma \rangle$, where $a = \sup_{x \in X} f(x)$, $b = \sup_{x \in X} g(x)$, $c = \sup_{x \in X} f(x) + g(x)$. Again, we need a normalization factor, here $a + b$.

— Upper: $c \times \_((f+g)/c) \dot{\leq} a \times \_(f/a) + b \times \_(g/b)$, up to normalization factor $a + b$, for all f, g as above.

— Linear: this just the conjunction of lower and upper.

— $P_{h \geq r}$: we would like to write $a + r \dot{\leq} (a-b) \times \_((a-h)/(a-b))$ where $a = \sup_{x \in X} h(x)$, $b = \inf_{x \in X} h(x)$, whenever h is not constant, or $a + r \dot{\leq} 0$ if h $(= a = b)$ is constant. This states that $a + r \leq m(z)(a-h)$, i.e., $r \leq \widehat{m(z)}(-h)$. Again, both sides of the first equation may fail to be in $[0,1]_\sigma$: if $0 \leq a + r \leq a - b$, we divide by the normalization factor $a - b$; if $a + r < 0$, then write nothing ($a + r \leq m(z)(a-h)$ always holds); if $a + r > a - b$, then write some inconsistent inequality such as $1 \dot{\leq} 0$ ($a + r \leq m(z)(a-h)$ never holds).

So Z^P is patch-closed in Z, hence stably compact by Proposition 5.5. But e and m form an homeomorphism between Z^P and $\mathsf{P}^P(X)$, so the latter is stably compact as well.

Next, define $\mathfrak{r} : \mathsf{P}^P(X) \to \mathbf{P}^P(X)$ by Scott's formula: $\mathfrak{r}(F)(f) = \sup_{h \text{ perfect} \ll f} F(h)$, using the fact that perfect maps form a basis (Proposition 7.11). This is continuous by Lemma 5.7. $\mathfrak{r}(F)$ is subnormalized whenever F is: for any constant a and any $h \in \langle X \to \mathbb{R}^+_\sigma \rangle$, $\mathfrak{r}(F)(a+h) = \sup_{g'' \text{ perfect} \ll a\chi_X + h} F(g'') = \sup_{\substack{g \text{ perfect} \ll a\chi_X \\ g' \text{ perfect} \ll h}} F(g+g')$ (since addition preserves and reflects $\ll$) $\leq \sup_{g' \text{ perfect} \ll h} F(a+g') \leq a + \mathfrak{r}(F)(h)$.

Next, $\mathfrak{r}(F)$ is also normalized whenever F is. $\mathfrak{r}(F)(a+h) = a + \mathfrak{r}(F)(h)$ is obvious when $a = 0$, so assume $a > 0$. Among the perfect maps $g \ll a\chi_X$, we find the maps of the form $a'\chi_X$ with $0 \leq a' < a$, by Lemma 7.20. Then $\mathfrak{r}(F)(a+h) = \sup_{\substack{g \text{ perfect} \ll a\chi_X \\ g' \text{ perfect} \ll h}} F(g+g') \geq \sup_{\substack{a' < a \\ g' \text{ perfect} \ll h}} F(a'+g') = \sup_{\substack{a' < a \\ g' \text{ perfect} \ll h}} (a' + F(g')) = a + \mathfrak{r}(F)(h)$.

We show that $\mathfrak{r}(F)$ is lower, resp. upper, resp. linear, whenever F is, by appealing again to the fact that addition preserves and reflects $\ll$.

Finally, if F is normalized and satisfies $P_{h \geq r}$, where h is perfect from X^{d} to $\mathbb{R}^+$, then we claim that $\mathfrak{r}(F)$ also satisfies $P_{h \geq r}$. We shall show this under the extra assumption that $\inf_{x \in X} h(x) > 0$: the general case reduces to it, since F satisfies $P_{h \geq r}$ iff F satisfies $P_{h+\delta \geq r-\delta}$ for any $\delta > 0$, using the fact that $\widehat{F}$ is normalized (Lemma 7.8), and since $\mathfrak{r}(F)$ satisfies $P_{h+\delta \geq r-\delta}$ iff it satisfies $P_{h \geq r}$, by a similar argument (recall from above that $\mathfrak{r}(F)$ is normalized, hence also $\widehat{\mathfrak{r}(F)}$). So assume $\inf_{x \in X} h(x) > 0$. Since F satisfies $P_{h \geq r}$, $\widehat{F}(-h) \geq r$, i.e., $F(a-h) \geq a + r$ for some large enough constant a, e.g., one such that $a > \sup_{x \in X} h(x)$. For all perfect maps $g \ll^{\mathsf{d}} h$, $\sup_{g' \text{ perfect} \ll a-g} F(g') \geq a + r$:

take indeed $g' = a - h$, since by Lemma 7.19 $a - h \ll a - g$. We have just shown that $\mathfrak{r}(F)(a-g) - a \geq r$, i.e., $\widehat{\mathfrak{r}(F)}(-g) \geq r$. Since g is perfect, $\mathfrak{r}(F)^\perp(g) \leq -r$. As this holds for every perfect map $g \ll^{\mathsf{d}} h$, and $\mathfrak{r}(F)^\perp$ is continuous, we also have $\mathfrak{r}(F)^\perp(h) \leq -r$. Since h is perfect, $\widehat{\mathfrak{r}(F)}(-h) \geq r$. So $\mathfrak{r}(F)$ also satisfies $P_{h \geq r}$.

Together with the inclusion $\mathfrak{s} : \mathbf{P}^P(X) \to \mathsf{P}^P(X)$, $\mathbf{P}^P(X)$ is a retract of $\mathsf{P}^P(X)$ by Lemma 5.10, and is therefore stably compact by Lawson's Lemma 5.12. In particular, $\mathbf{P}_{1\ wk}(X)$, $\bigtriangledown \mathbf{P}_{1\ wk}(X)$, $\bigtriangleup \mathbf{P}_{1\ wk}(X)$ and $\mathbf{P}^{\triangle}_{1\ wk}(X)$ are stably compact.

In any of these spaces Y of continuous normalized previsions, let $\langle h \geq r \rangle$ be the set of all $F \in Y$ such that $F^\perp(h) \leq -r$, or equivalently $\widehat{F}(-h) \geq r$, where h is perfect from X^{d} to $\mathbb{R}^+_\sigma$. This is the image by $\mathfrak{r}$ of the corresponding set of (not necessarily continuous) normalized previsions satisfying $P_{h \geq r}$. The above arguments then entail that, in any of these spaces Y, $\langle h \geq r \rangle$ is a stably compact subspace, hence a compact subset. It is clearly saturated. It follows that $\_^\perp$ is continuous from $\mathbf{P}_{1\ wk}(X)^{\mathsf{d}}$ to $\mathbf{P}_{1\ wk}(X^{\mathsf{d}})$—the cases of spaces of lower, upper, and linear previsions is similar, and omitted. Indeed, using Proposition 7.11, a subbasis of the weak topology on $\mathbf{P}_{1\ wk}(X^{\mathsf{d}})$ is given by subsets of the form $[h > r]$, where h is not just continuous, but perfect. The inverse image of $[h > r]$ by $\_^\perp$ is the complement of $\langle h \geq -r \rangle$, which is open in $\mathbf{P}_{1\ wk}(X)^{\mathsf{d}}$.

We now use an argument that is similar to the one of Proposition 6.8 to establish the converse result.

Let $\mathcal{Q}$ be any compact saturated subset of Y, where Y is any one of the above spaces of continuous normalized previsions. As a subset of $[\langle X \to \mathbb{R}^+ \rangle \to \overline{\mathbb{R}^+}]_{\mathsf{p}}$, $\mathcal{Q}$ is again compact. Write $\uparrow \mathcal{Q}$ the upward-closure of $\mathcal{Q}$ in $[\langle X \to \mathbb{R}^+ \rangle \to \overline{\mathbb{R}^+}]_{\mathsf{p}}$. This is compact saturated. Since $\langle X \to \mathbb{R}^+ \rangle$ is a continuous poset, Lemma 5.16 applies. so $\uparrow \mathcal{Q}$ is also compact saturated in $[\langle X \to \mathbb{R}^+ \rangle \to \overline{\mathbb{R}^+}]$ (i.e., with the Scott topology). The latter is a bc-domain, hence a continuous dcpo, so $\uparrow \mathcal{Q}$ can be written as filtered intersection of finitary compacts $\uparrow \mathcal{E}$, $\mathcal{E}$ finite subset of $[\langle X \to \mathbb{R}^+ \rangle \to \overline{\mathbb{R}^+}]$.

For any $F \in \mathcal{E}$, we claim that $\uparrow F$, the upward-closure of F in $[\langle X \to \mathbb{R}^+ \rangle \to \overline{\mathbb{R}^+}]$, equals $\bigcap_{g \text{ perfect}:X^{\mathsf{d}} \to \mathbb{R}^+_\sigma} \langle g \geq \widehat{F}(-g) \rangle^*$, where we write $\langle g \geq r \rangle^*$ for $\{F' \in [\langle X \to \mathbb{R}^+ \rangle \to \overline{\mathbb{R}^+}] \mid \widehat{F'}(-g) \geq r\}$. Indeed, if $F' \in \uparrow F$, then for every perfect map $g : X^{\mathsf{d}} \to \mathbb{R}^+_\sigma$, $-g$ is perfect from X to $\mathbb{R}_\sigma$, so $\widehat{F'}(-g) \geq \widehat{F}(-g)$. Conversely, if $F' \in \bigcap_{g \text{ perfect}:X^{\mathsf{d}} \to \mathbb{R}^+_\sigma} \langle g \geq \widehat{F}(-g) \rangle^*$, then $\widehat{F'}(-g) \geq \widehat{F}(-g)$ for all perfect maps $g : X^{\mathsf{d}} \to \mathbb{R}^+_\sigma$. This implies that $F'(g') \geq F(g')$ for all perfect maps $g' : X \to \mathbb{R}^+_\sigma$ (take $g = a - g'$ for some large enough constant a). Since the perfect maps form a basis (Proposition 7.11), and F and F' are continuous, $F' \in \uparrow F$.

So $\uparrow \mathcal{E}$ is a finite union of intersections of sets of the form $\langle g \geq r \rangle^*$. Therefore $\uparrow \mathcal{Q}$, as a filtered intersection of such sets $\uparrow \mathcal{E}$, is an intersection of finite unions of sets of the form $\langle g \geq r \rangle^*$. It is easy to see that $\mathcal{Q} = \uparrow \mathcal{Q} \cap Y$, and that $\langle g \geq r \rangle^* \cap Y = \langle g \geq r \rangle$, so that the cocompact topology on Y is exactly generated by the complements of the sets $\langle g \geq r \rangle$, where g ranges over the perfect maps from X^{d} to $\mathbb{R}^+$, $r \in \mathbb{R}$. Again using the fact that the weak topologies are generated by subsets of the form $[h > r]$, h perfect, we conclude that $\_^\perp$ is continuous from $\mathbf{P}_{1\ wk}(X^{\mathsf{d}})$ to $\mathbf{P}_{1\ wk}(X)^{\mathsf{d}}$, and similarly for spaces of lower, upper, and linear normalized previsions. ☐

Corollary 7.22 (Duality, Forks, Topological Version). Let X be stably compact.

$\mathbf{F}_{1\ wk}(X)$ is a stably compact space, and $\_^{\perp}$, as defined in Corollary 7.18, defines an involutive order-isomorphism from $\mathbf{F}_1(X)^{\mathsf{d}}$ to $\mathbf{F}_1(X^{\mathsf{d}})$.

Proof. Taking some notations from the proof of Theorem 7.21, $\mathbf{F}_{1\ wk}(X)$ arises as a retract of the subspace Fk of $ZZ = Z^{\text{“lower”, “normalized”}} \times Z^{\text{“upper”, “normalized”}}$ defined by the image of Walley's condition $F^-(h+h') \leq F^-(h) + F^+(h') \leq F^+(h+h')$ through the isomorphism defined by $e \times e$ and $m \times m$. That is, Fk is the set of those $(z^-, z^+) \in ZZ$ such that, for all non-identically zero $h, h' \in \langle X \to \mathbb{R}^+_\sigma \rangle$, with $a = \sup_{x \in X} h(x)$, $a' = \sup_{x\in X} h'(x)$, $(a+a') \times z^-_{(h+h')/(a+a')} \leq a z^-_{h/a} + a' z^+_{h'/a'} \leq (a+a') \times z^+_{(h+h')/(a+a')}$, and such that for every $h \in \langle X \to \mathbb{R}^+_\sigma \rangle$ with $\sup_{x\in X} h(x) = 1$, $z^-_h \leq z^+_h$. It is an easy exercise to show that Fk is patch-closed in ZZ, hence stably compact; this works as for Proposition 5.5. It follows that $\mathbf{F}_{1\ wk}(X)$ is stably compact, by Lawson's Lemma 5.12.

Now $\_^{\perp}$ maps normalized forks to normalized forks by Corollary 7.18. The rest of the Corollary is an easy consequence of Theorem 7.21. ☐

We finally observe that the duality $F \mapsto F^{\perp}$ on spaces of continuous normalized previsions extends the duality $\nu \mapsto \nu^{\perp}$ on spaces of continuous normalized games. Recall that one can define the Choquet integral $\oint_{x\in X} g(x) d\nu$ of a bounded continuous map $g : X \to \mathbb{R}$, not necessarily with non-negative values, as $\widehat{\alpha_{\mathcal{C}}(\nu)}(g)$, i.e., as $-a + \oint_{x\in X} a + g(x) d\nu$ for $a \geq -\inf_{x\in X} g(x)$.

Proposition 7.23. Let X be stably compact. For every continuous normalized prevision F on X, $\gamma_{\mathcal{C}}(F^{\perp}) = \gamma_{\mathcal{C}}(F)^{\perp}$. For every continuous normalized game ν on X, $\alpha_{\mathcal{C}}(\nu)^{\perp} = \alpha_{\mathcal{C}}(\nu^{\perp})$. In fact, for any perfect map $g : X \to \mathbb{R}^+$,

$$\oint_{x\in X^{\mathsf{d}}} -g(x) d\nu^{\perp} = -\oint_{x\in X} g(x) d\nu \qquad (2)$$

Proof. For every cocompact subset $X \setminus Q$ of X, $\gamma_{\mathcal{C}}(F)^{\perp}(X \setminus Q) = 1 - \inf_{U \text{ open} \supseteq Q} F(\chi_U)$. On the other hand, $\gamma_{\mathcal{C}}(F^{\perp})(X \setminus Q) = F^{\perp}(\chi_{X\setminus Q}) = \sup_{g \text{ perfect} \ll^{\mathsf{d}} \chi_{X\setminus Q}} -\widehat{F}(-g)$.

For any open subset U containing Q, by Lemma 7.10, there is a perfect map $g' : X \to [0,1]_\sigma$ such that $\chi_Q \leq g' \leq \chi_U$. Take $g = 1 - g'$: then $-\widehat{F}(-g) = -F(g') + 1 \geq -F(\chi_U) + 1$. Taking sups yields $\sup_{g \text{ perfect} \leq \chi_{X\setminus Q}} -\widehat{F}(-g) \geq \gamma_{\mathcal{C}}(F)^{\perp}(X \setminus Q)$. However, $\sup_{g \text{ perfect} \leq \chi_{X\setminus Q}} -\widehat{F}(-g) = \sup_{g \text{ perfect} \leq \chi_{X\setminus Q}} F^{\perp}(g) = F^{\perp}(\chi_{X\setminus Q}) = \gamma_{\mathcal{C}}(F^{\perp})(X \setminus Q)$, since $F^{\perp}$ is continuous.

Conversely, let U' be the open subset of X^{d} defined as $X \setminus Q$. Since X^{d} is locally compact, U' is the directed union of all interiors (in X^{d}), $int^{\mathsf{d}}(Q')$, of compact saturated subsets Q' of X^{d} such that $Q' \subseteq U'$. In particular, $\chi_{U'} = \chi_{X\setminus Q}$ is the sup of the directed family of all $\chi_{int^{\mathsf{d}}(Q')}$, Q' as above. So, for each perfect map $g : X^{\mathsf{d}} \to \mathbb{R}^+_\sigma$ such that $g \ll^{\mathsf{d}} \chi_{X\setminus Q}$, there is a $Q' \subseteq U'$ as above such that $g \leq \chi_{int^{\mathsf{d}}(Q')} \subseteq \chi_{Q'}$. Let U be the complement of Q': we have found an open U of X containing Q, and such that $g \leq \chi_{X\setminus U}$. Then $-\widehat{F}(-g) \leq 1 - F(\chi_U)$. Taking sups, $\gamma_{\mathcal{C}}(F^{\perp})(X \setminus Q) \leq \gamma_{\mathcal{C}}(F)^{\perp}(X \setminus Q)$, whence the equality.

We now prove (2). On the one hand, letting $a \geq \sup_{x \in X} g(x)$,

$$\begin{aligned} -\oint_{x \in X^{\mathsf{d}}} -g(x) d\nu^{\perp} &= a - \oint_{x \in X^{\mathsf{d}}} (a - g(x)) d\nu^{\perp} \\ &= a - \int_0^a \nu^{\perp}((a-g)^{-1}(t, +\infty)) dt \\ &= a - \int_0^a \nu^{\perp}(X \setminus g^{-1}[a-t, +\infty)) dt = \int_0^a \nu^{\dagger}(g^{-1}[a-t, +\infty)) dt \end{aligned}$$

On the other hand,

$$\oint_{x \in X} g(x) d\nu = \int_0^a \nu(g^{-1}(t, +\infty)) dt = \int_0^a \nu(g^{-1}(a-t, +\infty)) dt$$

Observe that $g^{-1}(a-t, +\infty) \subseteq g^{-1}[a-t, +\infty)$. So every open subset U containing $g^{-1}[a-t, +\infty)$ also contains $g^{-1}(a-t, +\infty)$, whence $\nu(g^{-1}(a-t, +\infty)) \leq \nu^{\dagger}(g^{-1}[a-t, +\infty))$. So $\oint_{x \in X} g(x) d\nu \leq -\oint_{x \in X^{\mathsf{d}}} -g(x) d\nu^{\perp}$. Conversely, for every $\epsilon > 0$, $g^{-1}[a-t, +\infty)$ is contained in the open $g^{-1}(a-t-\epsilon, +\infty)$, so $\nu^{\dagger}(g^{-1}[a-t, +\infty)) \leq \nu(g^{-1}(a-t-\epsilon, +\infty))$. We deduce that:

$$\begin{aligned} -\oint_{x \in X^{\mathsf{d}}} -g(x) d\nu^{\perp} &\leq \int_0^a \nu(g^{-1}(a-t-\epsilon, +\infty)) dt \\ &= \int_0^{a-\epsilon} \nu(g^{-1}(a-t-\epsilon, +\infty)) dt + \epsilon\nu(X) \\ &= \int_{\epsilon}^a \nu(g^{-1}(a-t, +\infty)) dt + \epsilon\nu(X) \leq \oint_{x \in X} g(x) d\nu + \epsilon\nu(X) \end{aligned}$$

As $\epsilon > 0$ is arbitrary, $-\oint_{x \in X^{\mathsf{d}}} -g(x) d\nu^{\perp} \leq \oint_{x \in X} g(x) d\nu$, from which (2) follows.

We now realize that the left-hand side of (2) is $\widehat{\alpha_{\mathcal{C}}(\nu^{\perp})}(-g) = -\alpha_{\mathcal{C}}(\nu^{\perp})^{\perp}(g)$, while the right-hand side is $-\alpha_{\mathcal{C}}(\nu)(g)$. So $\alpha_{\mathcal{C}}(\nu^{\perp})^{\perp}$ and $\alpha_{\mathcal{C}}(\nu)$ coincide on all perfect maps $g : X \to \mathbb{R}^+$. By Proposition 7.11 they are the same prevision. Replacing ν by $\nu^{\perp}$ and using the fact that $\nu^{\perp\perp} = \nu$, we obtain $\alpha_{\mathcal{C}}(\nu)^{\perp} = \alpha_{\mathcal{C}}(\nu^{\perp})$. ☐

8. Conclusion

We hope to have demonstrated that convex-concave duality is a beautiful family of dualities that extend de Groot duality to various domains of non-deterministic, probabilistic, and mixed choice. This uncovers a hidden symmetry in powerdomains, whereby angels and demons trade places, while nature remains intact.

Acknowledgments

We thank Martín Escardó for the alternative proof of Theorem 3.1 and its extensions, Klaus Keimel for directing me to Ben Cohen's work, and both Klaus Keimel and Jimmie Lawson for fruitful comments. We also thank the anonymous referees for their extraordinarily detailed and deep comments. One of the referees found a crucial mistake in a

former version of Theorem 6.18, and both found some other mistakes as well. Both referees suggested a number of alternative proof arguments, among other things. I have mentioned some of them explicitly in the text. The idea of using approximations by perfect maps in Section 7 is also due to one of the anonymous referees, resulting in a considerable simplification of my original argument.

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