Some mathematical problems in domain theory

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I want to address two quite different aspects. Firstly, I want to discuss partial orders $\leq_C$ on a finite dimensional vector space V induced by a closed generating pointed cone C. V becomes a bicontinuous (= continuous and dually continuous) poset and the way-below relation coincides with the dual of the dual way-below relation. One may restrict the order to appropriate subsets like the cone C itself or its opposite $-C$; also the case of a subcone D of C seems of interest. The way-below relation on $(D, \leq_C)$ seems to be closely related to the geometry of the cone C. Almost nothing is known about the domain theoretical properties. When do we get FS-domains or retracts of bifinite domains? There are two old open questions related to these orders.

It is not so difficult to see that $-C$ augmented by a bottom element is an FS-domain with respect to the order $\leq_C$. **Is $C_\perp$ a retract of a bifinite domain?** In the special case of an ice-cream cone C in $\mathbb{R}^3$, Lawson and Jung conjectured that $-C_\perp$ is not a retract of a bifinite domain. But the problem whether every FS-domain is a retract of a bifinite one is still open.

For a finite poset P, a cone C in the vector space $\mathbb{R}^P$ is given by the linear inequalities

$$\sum_{p \in U} x_p \geq 0$$

where U ranges over the collection of upper sets in P. The associated order induces the stochastic order on the probabilistic and the subprobabilistic powerdomain over P. **It is an old problem whether these powerdomains are FS-domains or even retracts of bifinite domains**. The only results known are due to Jung and Tix and they concern only finite trees and root systems.

Similar partial orders as those induced by cones in vector spaces occur on smooth manifolds that are equipped with a cone field, i.e., an assignment of a closed generating cone $C(x)$ in the tangent space $V(x)$ to each point x on the manifold. Under appropriate hypotheses on the cone field one obtains a partial order with the same properties as above by considering the closure of the relation $x \prec y$ iff there is a smooth curve from x to y the direction (more precisely, the derivative) of which in is in the interior of the cone $C(x)$ in each point of the curve. A particular instance of this situation has been considered by K.Martin and P. Panangaden on spacetime manifolds.

Secondly we want to address problems that arise in connection with the probabilistic powerdomain. When characterizing algebras of monads of (sub-probability) measures one meets abstract topological convex sets and cones. An abstract cone is a set C with an addition and multiplication with non-negative real numbers $(r, x) \mapsto rx$ which satisfy the same equational laws as in vector spaces. Not all abstract cones are embeddable in vector spaces, as every join semilattice with bottom can be considered to be an abstract cone. But if C is endowed with a Hausdorff topology such that addition and scalar multiplication are continuous, then C is embeddable as a cone in a real vector space V. If, in addition, the topology is locally compact, then V can be endowed with a vector space topology in such a way that the embedding is topological, too, as Lawson and Madison have shown. We can extend this result to locally compact Hausdorff cones with a closed compatible order. Then V has a vector space ordering extending that of C. This result can be used to show that the algebras of the monad of (sub-)probability measures with the stochastic ordering over compact ordered spaces are the compact convex sets (with 0) in ordered locally convex topological vector spaces.

We consider this result as a step towards a solution of the problem: **Characterise the algebras of the (sub-)probabilistic powerdomain over stably compact spaces**.