# BITOPOLOGICAL DUALITY FOR DISTRIBUTIVE LATTICES AND HEYTING ALGEBRAS

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It is widely considered that the beginning of duality theory was Stone's groundbreaking work in the mid 30ies on the dual equivalence of the category **Bool** of Boolean algebras and Boolean algebra homomorphism and the category **Stone** of compact Hausdorff zero-dimensional spaces, which became known as Stone spaces, and continuous functions. In 1937 Stone [7] extended this to the dual equivalence of the category **DLat** of bounded distributive lattices and bounded lattice homomorphisms and the category **Spec** of what later became known as spectral spaces and spectral maps. Spectral spaces provide a generalization of Stone spaces. Unlike Stone spaces, spectral spaces are not Hausdorff (not even $T_1$), and as a result, are more difficult to work with. In 1970 Priestley [6] described another dual category of **DLat** by means of special ordered Stone spaces, which became known as Priestley spaces, thus establishing that **DLat** is also dually equivalent to the category **Pries** of Priestley spaces and continuous order-preserving maps. Since **DLat** is dually equivalent to both **Spec** and **Pries**, it follows that the categories **Spec** and **Pries** are equivalent. In fact, more is true: as shown by Cornish [1] (see also Fleisher [4]), **Spec** is actually isomorphic to **Pries**. The advantage of Priestley spaces is that they are easier to work with than spectral spaces. As a result, Priestley's duality became rather popular, and most dualities for distributive lattices with operators have been performed in terms of Priestley spaces. Here we only mention Esakia's duality for Heyting algebras, co-Heyting algebras, and bi-Heyting algebras [2, 3], which is a restricted version of Priestley's duality.

Another way to represent distributive lattices is by means of bitopological spaces, as demonstrated by Jung and Moshier [5]. In this paper we provide an explicit axiomatization of the class of bitopological spaces obtained this way. We call these spaces *pairwise Stone spaces*. On the one hand, pairwise Stone spaces provide a natural generalization of Stone spaces as each of the three conditions defining a Stone space naturally generalizes to the bitopological setting: compact becomes pairwise compact, Hausdorff – pairwise Hausdorff, and zero-dimensional – pairwise zero-dimensional. On the other hand, pairwise Stone spaces provide a natural medium in moving from Priestley spaces to spectral spaces and backwards, thus Cornish's isomorphism of **Pries** and **Spec** can be established more naturally by first showing that **Pries** is isomorphic to the category **PStone** of pairwise Stone spaces and bicontinuous maps, and then showing that **PStone** is isomorphic to **Spec**. Thirdly, the signature of pairwise Stone spaces naturally carries the symmetry present in Priestley spaces (and distributive lattices), but hidden in spectral spaces. Moreover, the proof that **DLat** is dually equivalent to **PStone** is simpler than the existing proofs of the dual equivalence of **DLat** with **Spec** and **Pries**.

One of the advantages of Priestley's duality is that many algebraic concepts important for the study of distributive lattices can be easily described by means of Priestley spaces. In addition, we show that they have a natural dual description by means of pairwise Stone spaces. We give their dual description by means of spectral spaces, which at times is less transparent than the order topological and bitopological descriptions. We also introduce the subcategories of **PStone** and **Spec**, which are isomorphic to the category **Esa** of Esakia spaces and dually equivalent to the category **Heyt** of Heyting algebras. This provides an alternative of Esakia's duality in the setting of bitopological spaces and spectral spaces. In addition, we establish similar dual equivalences for the categories of co-Heyting algebras and bi-Heyting algebras.

## References


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