Equideductive Logic and CCCs with Subspaces

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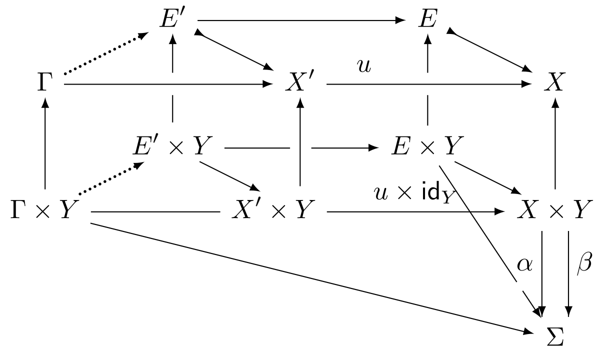
In any cartesian closed category with equalisers, the logic of regular monos (maps that arise as equalisers) of course has conjunctions. But also, if $\mathfrak{p}(x)$ represents a regular mono into X and $\alpha, \beta : X \times Y \rightrightarrows \Sigma$ are any maps then there is a regular mono into Y represented by $\mathfrak{q}(y) = \forall x.\mathfrak{p}(x) \Rightarrow \alpha(x,y) = \beta(x,y) : \Sigma$. Categorically, $\mathfrak{q}(y)$ is defined by a kind of partial product:



This apparently rather feeble logic is nevertheless interesting because

- it is how we reason with proofs of equations in algebra, ie treating judgements that one equation follows from others, proof rules about such judgements (such as induction schemes), etc., as arbitrarily nestable implications;
- it may be interpreted in the category of sober topological spaces, and possibly also in the category of locales;
- together with the lattice structure on the Sierpinski space, it is the logic that is needed to form open, closed, compact and overt subspaces in ASD;
- it provides the abstract (type-theoretic) basis on which construct a cartesian closed category with equalisers similar to Scott's equilogical spaces.

Given some *urtypes* (playing the role of unions of algebraic lattices) and combinators on them, an *object* X is a triple $(A, \mathfrak{p}, \mathfrak{q})$ where A is an urtype, $\mathfrak{p}$ is a predicate in this logic on Σ^A, and $\mathfrak{q}$ one on A, for which

$$\phi, \psi : \Sigma^A,\ \mathfrak{p}(\phi),\ (\forall a{:}A.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a)\ \vdash\ \mathfrak{p}(\psi).$$

A *morphism* $M : X \equiv (A, \mathfrak{p}, \mathfrak{q}) \to Y \equiv (B, \mathfrak{r}, \mathfrak{s})$ is an urterm $M : (A \to \Sigma) \to B \to \Sigma$ such that

$$\phi : \Sigma^A,\ \mathfrak{p}(\phi)\ \vdash\ \mathfrak{r}(M\phi)$$

and $\qquad \phi, \psi : \Sigma^A, \quad \mathfrak{p}(\phi), \quad \forall a.\, \mathfrak{q}(a) \Rightarrow \phi a = \psi a \quad \vdash \quad \forall b.\, \mathfrak{s}(b) \Rightarrow M\phi b = M\psi b,$

where $M_1 = M_2$ if $\qquad \phi : \Sigma^A,\ b : B,\ \mathfrak{p}(\phi),\ \mathfrak{s}(b)\ \vdash\ M_1\phi b = M_2\phi b.$

This gives a cartesian closed category with finite limits and colimits and some exactness properties.