A CONTINUOUS DCPO REPRESENTATION OF REGULAR FORMAL TOPOLOGIES

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Given a formal topology $\mathcal{X} = (X, \lhd, \leq, \mathrm{Pos})$ there is a natural *well-inside* relation $\ll$ on X defined by

$$a \ll b \equiv X \lhd\, a^{\perp} \cup \{b\},$$

where $a^{\perp} = \{c \in X \mid a_{\leq} \cap c_{\leq} \lhd \emptyset\}$ and $x_{\leq} = \{y \in X \mid y \leq x\}$. The formal topology $\mathcal{X}$ is said to be *regular* if $b \lhd \{a \in X \mid a \ll b\}$ for all $b \in X$. Furthermore, we say that $\ll$ is dense on X if $a \ll b$ implies the existence of a $c \in X$ with $a \ll c \ll b$.

We show that when $\mathcal{X}$ is regular, $(X, \leq)$ is a *consistently complete lower semi-lattice pre-order* (CLSP) and $\ll$ is dense on X, there is a continuous dcpo D such that $Pt(\mathcal{X}) \subseteq D$ and $Pt(\mathcal{X})$ is equipped with the Scott topology induced by D. That is, $(D, Pt(\mathcal{X}), id)$ is a continuous dcpo representation of the formal space $Pt(\mathcal{X})$. The dcpo D is given as the class of points of a formal topology extending $\mathcal{X}$. We also prove a lifting result that says that each continuous morphism $F : \mathcal{X} \to \mathcal{Y}$ between such formal topologies induces a Scott continuous function $f : D_{\mathcal{X}} \to D_{\mathcal{Y}}$, where $D_{\mathcal{X}}$ and $D_{\mathcal{Y}}$ are the continuous dcpo representations of $\mathcal{X}$ and $\mathcal{Y}$ respectively, satisfying $f\mid_{Pt(\mathcal{X})} = Pt(F)$.

The formal reals, $\mathcal{R}$, can be presented as a regular CLSP topology with dense well-inside relation, and therefore has a continuous dcpo representation of the above sort. Basic opens of $\mathcal{R}$ are pairs of rational numbers (p, q) with $p < q$. Points of this topology will include not only formal reals and closed intervals of formal reals but also nonconstructive elements. An interesting feature of this representation is then that liftings of continuous functions on the formal reals can be applied to elements of the later sorts. It is also shown that the arithmetic of points in $D_{\mathcal{R}}$ coincide with interval arithmetic when restricted to "interval points" $I[x, y] = \{(p, q) \mid p < x \leq y < q\}$ for formal reals x, y. Moreover, continuos functions $f : \mathbb{R} \to \mathbb{R}$ lift to continuous functions $\widehat{f} : D_{\mathcal{R}} \to D_{\mathcal{R}}$ taking interval points to interval points representing the functional image in the sense that

$$\widehat{f}(I[x, y]) = I\left[\inf_{z \in [x,y]} f(z), \sup_{z \in [x,y]} f(z)\right].$$

In [1], the authors introduce a domain-theoretic framework for differential calculus. They define an operation D on continuous functions $f : \mathbb{IR} \to \mathbb{IR}$ yielding a continuous function $D(f)$, such that $D(If) = I(f')$ for differentiable functions f (If being the lifting of f to $\mathbb{IR}$ - the domain of closed real intervals ordered by reverse inclusion). We have investigated how well this framework transfers to the dcpo $D_{\mathcal{R}}$ and have shown that the analogue of the operation D satisfies $D(\widehat{f}\,) = \widehat{f'}$ for (Bishop)-differentiable functions f. All results are constructive in the sense that we use constructive logic. This is work in progress.

References

[1] Abbas Edalat, André Lieutier, *Domain theory and differential calculus (functions of one variable)*, Mathematical Structures in Computer Science, Volume 14, Issue 06, Dec 2004, pp 771-802.