Two Probabilistic Powerdomains for Topological Domain Theory

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Abstract

In previous research by the authors and Schröder, Topological Domain Theory has been proposed as a versatile framework for denotational semantics for programming languages, overcoming some of the difficulties present in Classical Domain Theory, such as combining function types, polymorphism and computational effects in a single semantic model. In this work, we present two approaches to modelling probabilistic computations in Topological Domain Theory.

The central objects of Topological Domain Theory are *topological predomains*, which are topological spaces that are both a quotient of a countably-based space and a monotone convergence space. The category of topological predomains is cartesian closed with countable limits and colimits, supports the usual constructions of domain theory, and contains all ω-continuous dcpos.

Our first approach to model probabilistic compuations within topological predomains is given by a *free convex space* construction, fitting into the theory of modelling computational effects via free algebras for equational theories, as proposed by Plotkin and Power. Here we get the following result:

Theorem 1. *The free convex space construction on topological predomains extends the classical probabilistic powerdomain on ω-continuous dcppos.*

We remark that by the classical probabilistic powerdomain we refer to the dcpo of probability valuations rather than subprobability valuations as in the original work by Jones and Plotkin.

Our second approach is given by an *observationally induced probabilistic powerdomain*, following recent work of Schröder and Simpson. Here the key idea is to use the unit interval equipped with the Scott-topology as an object of observation values to determine an algebraic structure and a completeness property on other topological predomains. This is done much in the same way in which Sierpinski space can be used to determine a completeness property characterising sober topological spaces. For the observational approach we get the following result:

Theorem 2. *The free observationally induced probabilistic powerdomain on a countably-based topological predomain is given by the classical probabilistic powerspace.*

By the classcial probabilistic powerspace we refer to the set of probability valuations equipped with the *weak topology* (see Jung's work on modelling probabilistic features on stably compact spaces), or equivalently the *pointwise topology* (see Heckmann's work on spaces of valuations).

It is well-known that for continuous dcpos, the constructions of the classical probabilistic powerspace and the classical probabilistic powerdomain coincide. Thus, on ω-continuous dcppos our two approaches agree (and indeed they yield the classical construction). However, they do not agree on all countably-based spaces: for instance the free convex predomain over the space of natural numbers does not carry the pointwise topology - it's topology is in fact strictly finer.

Furthermore, we note that Theorem 2 cannot be generalised beyond countably-based spaces. Using a space given by Grunhage and Streicher as a counterexample showing that Topological Domain Theory is not closed under sobrification, we can show the following:

Theorem 3. *The category of topological predomains is not closed under the classical probabilistic powerspace construction.*