Continuous V-categories

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One of the nice features of domain theory is the strong interaction between order-theoretic, topological and algebraic ideas. For instance, continuous lattices [4] can be described as ordered sets with certain completeness properties, as injective topological T_0-spaces with respect to embedding, or as Eilenberg–Moore algebras for the filter monad on Set. In this talk we will consider an approach to domain theory using enriched category theory.

Since F.W. Lawvere's famous 1973 paper [3] it is well-known that both ordered sets and metric spaces can be viewed as V-enriched categories: the former ones for the quantale $\mathsf{V} = 2$, the latter ones for the quantale $\mathsf{V} = [0, \infty]$. Thanks to M. Barr [1] we know that topological spaces can be presented as categories as well, by interpreting the convergence relation $\mathfrak{x} \to x$ between ultrafilters and points of a topological space X as arrows in X. Of course, we have to make here the concession that the domain of a morphism is not just an object but (for instance) an ultrafilter of objects. In order to unify both of the above-mentioned approaches, we introduce the notion of a $(\mathbb{T}, \mathsf{V})$-category, for a Set-monad $\mathbb{T}$ and a commutative quantale $(\mathsf{V}, \otimes, k)$. After developing basic category theory for $(\mathbb{T}, \mathsf{V})$-categories (Yoneda lemma, presheaf categories, Kan extension,...), we introduce cocomplete $(\mathbb{T}, \mathsf{V})$-categories with respect to a class Φ of distributors, and characterise Φ-cocomplete $(\mathbb{T}, \mathsf{V})$-categories as precisely the injective ones with respect to fully faithful Φ-dense functors. Furthermore, we show that the category of separated and Φ-cocomplete $(\mathbb{T}, \mathsf{V})$-categories and Φ-cocontinuous functors is monadic over $(\mathbb{T}, \mathsf{V})$-Cat, V-Cat and Set respectively. As a consequence, we obtain a V-enriched equivalent to the filter monad, whose algebras might deserve to be called continuous V-categories. More general, for a suitable choice of the class Φ of distributors, we recover several of the categories of semantic domains studied in [2] together with possible metric and V-enriched counterparts.

The talk is based on several (joint) papers, please consult also `http://www.mat.ua.pt/pessoais/dirk`

References

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[3] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano, 43 (1973), pp. 135–166 (1974). Also in: Repr. Theory Appl. Categ. 1:1–37 (electronic), 2002.

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