Equational logic for higher-order abstract syntax

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Syntax with variable binders cannot be captured as an initial algebra in the usual way. But Fiore, Plotkin and Turi showed that this is possible if one moves from algebras for a functor on Set to algebras for a functor on a suitable presheaf category. In particular, they showed that lambda-terms up to alpha-equivalence form an initial algebra for a functor. These functors generalize the notion of a signature, but a notion of equational theory for these algebras is missing in their work.

We will give such an equational theory as an application of a more general result concerning functors having finitary presentations. The notion of a finitary presentation for a functor generalizes the notion of a presentation for an algebra and was introduced by Kurz and Bonsangue in order to describe functors corresponding to modal logics for coalgebras. Logics for T-coalgebras (where T is an endofunctor on Set) are suitably described by endofunctors L on the category of Boolean algebras. Syntactically, L specifies an extension of Boolean propositional logic by modal operators and axioms. Semantically, L gives a logical description of the 'transition type' T of the coalgebras. Functors appearing in this way have a finitary presentation. An important result concerning this class of functors states that algebras for such a functor form an equationally definable class. We will use a generalization of this theorem to the case of many-sorted varieties.

Canonical representatives for λ-terms up to α-equivalence can be obtained in different ways, for example, using the method of De Bruijn levels: normal forms up to α-equivalence are obtained by specifying well-formedness rules for λ-terms within a context. The appropriate notion to encompass contexts and the operations allowed on them is the category $\mathbb{F}$ of finite cardinals and functions. Fiore, Plotkin and Turi constructed an endofunctor L on the presheaf category $\mathsf{Set}^{\mathbb{F}}$ such that for any presheaf (of variables) V, then the λ-terms over V, up α-equivalence, form the free L-algebra over V.

Let us notice that category of presheaves is a many-sorted variety. We obtain the algebraic structure for the equivalence classes of λ-terms by giving an equational presentation for $\mathsf{Alg}(L)$. First we find an equational presentation for the many-sorted variety $\mathsf{Set}^{\mathbb{F}}$. Then we show that L has a finitary presentation. Now we can apply the theorem stated above and obtain an equational presentation for $\mathsf{Alg}(L)$.

Our work differs from the recent work of Fiore and Hur in that our notion of equational theory is the standard one from universal algebra.